Generalized Differential Transformation Method for Solving system of Non linear Volterra integro-differential equations of fractional order

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A b s t r a c t

In this paper, the technique of modified Generalized Differential Transformation Method (GDTM) is used to solve a system of Non linear integro-differential equations with initial conditions. Moreover, a particular example has been discussed in three different cases to show reliability and the performance of the modified method. The fractional derivative is considered in the Caputo sense .The approximate solutions are calculated in the form of a convergent series, numerical results explain that this approach is trouble-free to put into practice and correct when applied to systems integro-differential equations.

Key Words: Integro-Differential equations, Fractional calculus, Generalized Differential Transformation method, system of Non linear Volterra integro-differential equations .

1. Fractional Calculus

In fractional calculus (a subject of mathematics which grows out of the customary definitions of the calculus integral and integral operators in which the same by fractional exponents in an outgrowth of exponents with integral value),the initial value problems plays an important role in many fields[21]. This type of problems has been taken a lot of interest; so most of the mathematical theory valid to the study of fractional calculus was developed preceding to the turn of the 20th century. However it is in the past 100 years that the most interesting leaps in engineering and scientific application have been found. The mathematics has in some cases had to change to meet the requirements of physical actuality[19], [21].

There is some basic definitions in fractional calculus mention the most important :

Definition (1.1): (Caputo Fractional Derivatives D_C^α),[4],[10],[19]:

Let $f(t) \in C_\mu^n$ [17] that is defined on the closed interval [a,b], the Caputo fractional derivative of order α> 0 of f is defined by:

$$D_c^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t \dfrac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \,, & n-1 < \alpha < n\,, \quad n \in N \\ \dfrac{d^n}{dt^n} f(t), & \alpha = n, \quad n \in N \end{cases} \qquad (1.1)$$

Definition (1.2): (Riemann- Liouville Fractional Integrals),[16],[17]:

Let $f(t) \in C_\mu^n$ that is defined on the closed interval [a,b], Riemann-Liouville Fractional integral of order α> 0 of f is defined by: $J^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1} d\tau$ (1.2)

Definition(1.3)(Gamma function),[12], [14], [19]

The complete gamma function $\Gamma(t)$ is also known as generalized factorial function. It is defined by using the following integral:

$\Gamma(t) = \int_0^\infty S^{t-1} e^{-S} dS\,, \quad t > 0$, S any variable (1.3)

(1.5)(Properties of Gamma function), [14],[19]:

(1) $\Gamma(t+1) = t\Gamma(t) \quad t > 0$

(2) $\Gamma(t) = (t-1)!$t is positive integer, convention: 0! = 1

(3)$\Gamma\frac{1}{2}=\sqrt{\pi}$

2. Differential transform DT [2],[3],[8],[18]:

There is many numerical methods that have been adopted to resolve this type of problems such as Adomian decomposition method (ADM), variational iteration method (VIM), homotopy analysis method(HAM), homotopy perturbation method (HPM) and differential transform method(DTM),[13],[15]. Generalized Differential transform (GDT) has taken the shape of an important and convenient tool. In(1980) G.E. Pukhov used differential transform in numerical methods to solve fractional differential equations for the first time[8],[9]. The concept of the differential transform method was first widely proposed by Zhou (1986)], who solved linear and nonlinear initial value problems in electric circuit analysis..[22]. Since then, (DTM) was success-fully applied for a large variety of problems. In (2008) Erturk and Momani proposed (DTM) as efficiency tool to solve systems of fractional differential equations [5],[11]. After this many researchers used (DTM) till Taghvafard and Erjaee in (2011) solved systems of singular Volterra integro-differential equations of convolution type with DT [6]. One can define the differential transform method (DTM) as a semi-analytical-numerical method that uses Taylor series for the solution of differential equations. This method constructs approximate solution in the form of a polynomial and finally gives a series solution. It is different from the traditional high order Taylor's series, as, it requires a long time in calculation and needs computation of the necessary derivatives of the data functions

2.1 Fractional Differential transform [5],[7]:

Fractional Differential transform can be defined as:

$$F(k)=\begin{cases} \frac{k}{\alpha}\in Z^{+}, \frac{1}{\left(\frac{K}{\alpha}\right)!}\left[\frac{d^{\frac{k}{\alpha}}f(x)}{dx^{k/\alpha}}\right]_{x=x_0} \text{for } k=0,1,\dots,(q\alpha-1) \\ \frac{k}{\alpha}\notin Z^{+}, 0 \end{cases} \qquad (1.4)$$

Where α is the order of fractional derivative.

And We define the generalized differential transform of the kth derivative of function f(t) in one variable as follows,[6]: $F(k)=\frac{1}{\Gamma(\alpha k+1)}[\left(D_{t_\circ}^{\alpha}\right)^{k} f(t)]_{t=t_\circ}$ (1.5)

where $(D_{t_\circ}^{\alpha})^{K}=D_{t_\circ}^{\alpha}.D_{t_\circ}^{\alpha}\dots\dots D_{t_\circ}^{\alpha}$, k-times and the differential inverse transform of F(k)is defined as follows: $f(t)=\sum_{k=0}^{\infty}F_{\alpha}(k)\left(t-t_\circ\right)^{\alpha k}$ (1.6)

2.2 Properties of GDTM ,[5], [6], [7], [20]:

1-If $f(t)=g(t)\pm h(t)$, then $F(k)=G(k)\pm H(k)$.

2-If f(t) =ag(t), then $F(k)=aG(k)$, where a is a constant.

3-If f(t) =g(t)h(t), then$F(k)=\sum_{l=0}^{k}G(l)\ H(k-l)$

4-If f(t)=$g_1(t)g_2(t),\dots\dots..,g_{n-1}(t)g_n(t)$, then

$$f(x)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-1}=0}^{k_{n-1}}\dots\dots\sum_{k_{n-1}=0}^{k}\sum_{k_{n-1}=0}^{k}G_1(k_1)G_n(k_2-k_1)\dots\dots G_{n-1}(k_{n-1}-k_{n-2})G_n(k-k_{n-1})$$

5-If f(t)= $D_{t_\circ}^{\alpha}g(t)$, then $F(k)=\frac{\Gamma(\alpha k+1)+1}{\Gamma(\alpha k+1)}G(k+1)$

6-If $f(t)=(t-t_{\circ})^{\beta}$, then $F(k)=\delta\left(k-\frac{\beta}{\alpha}\right)$,where $\delta(k)=\begin{cases}1 \text{ if } k=0\\ 0 \text{ if } k\neq 0\end{cases}$

7-If $f(t)=\int_{t_{\circ}}^{t} g(t)dt$, then $F(k)=\frac{G\left(k-\frac{1}{\alpha}\right)}{\alpha k}$ where $k\geq\frac{1}{\alpha}$

8-If $f(t)=g(t)\int_{t_{\circ}}^{t} h(t)dt$ then $F(k)=\sum_{k_1=\frac{1}{\alpha}}^{k}\frac{H\left(k-\frac{1}{\alpha}\right)}{\alpha k_1}G(k-k_1)$ where $k\geq\frac{1}{\alpha}$

9-If $f(t)=\int_{t_{\circ}}^{t} h_1(t)h_2(t)\ldots\ldots\ldots h_{n-1}(t)h_n(t)dt$,then

$F(k)=\frac{1}{\alpha k}\sum_{k_{n-1}=0}^{k-\frac{1}{\alpha}}\sum_{k_{n-2}}^{k_{n-1}}\ldots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}H_1(k_1)H_2(k_2-k_1)..H_{n-1}\left(k_{n-1}-k_n-\frac{1}{\alpha}\right], k\geq\frac{1}{\alpha}$.

10- If $f(t)=[g_1(t)g_2(t)\ldots\ldots\ldots\ldots g_{m-1}(t)g_m(t)]\int_{t_{\circ}}^{t} h_1(t)h_2(t)\ldots.h_{n-1}(t)h_n(t)dt$,

then $F(k)=\sum_{k_i}^{k}\frac{1}{\alpha k_1}\sum_{j_{n-1}=0}^{k_1-\frac{1}{\alpha}}\sum_{j_2=0}^{j_3}\sum_{j_1=0}^{j_2}\sum_{i_1=0}^{k-k_1}\sum_{i_{m-2}=0}^{i_{m-1}}G_1(i_1)\,G_2(i_2-i_1)\ldots G_{m-1}(i_{m-1}-$
$i_{m-2})G_m(k-i_{m-1}-k_1)\times H_1(j_1)H_2(j_2-j_1)\ldots..H_{n-1}(j_{n-1}-j_{n-2})H_n(k_1-j_{n-1}-\frac{1}{\alpha})$,
where $k\geq 1/\alpha$.

11-For special functions that may used

- If $f(t)=e^{\lambda t}$ then $F(k)=\frac{\lambda^k}{k!}$
- If $f(t)=\sin(at+\beta)$ then $F(k)=\frac{a^k}{k!}\sin\frac{\pi k}{2}+\beta$
- If $f(t)=\cos(at+\beta)$ then $F(k)=\frac{a^k}{k!}\cos\frac{\pi k}{2}+\beta$

In this paper, the differential transformation method is modified to solve a system of Nonlinear integro-differential equations of fractional order with initial conditions. A a small number of searchers engross with fractional systems ,so the propose technique provide a good results for non linear system of fractional integro-differential equations.

3.Solving Non Linear Volterra integro- differential equations of fractional order (NL-FVIDE **Using GDTM Technique. ,[6],[7],[8]:**

The technique that we used is the differential transform method (DTM),which is based on Taylor series expansion. It is introduced by Zhou[60] in a study about electrical circuits.

Consider the NL-FVIDE. $D^{\beta}u(t) = f(t) + \lambda\int_0^t K(t,x,u)u^i(x)dx$ (3.1)

with initial condition u(0)= a , $0< \beta \le 1$,$\lambda \in \mathbb{R}$, i=1,…,n ,$n \in \mathbb{Z}^+$

$D^{\beta}u(t)$denotes the Caputo fractional derivative of order β for u(t). f(t) is continuous function.

To solve the equation (3.1) using GDTM, one can take the differential transform for both sides of equation (3.1). According to GDTM' s properties in (2.2),the terms of equation (3.1) can be transform as following:

1- $D^{\beta}u(t)$ transformed to $\frac{\Gamma(\alpha k+\beta+1)}{\Gamma(\alpha k+1)}U(k+\frac{\beta}{\alpha})$

2- f(t)transformed toF(k)

3-$\lambda\int_0^t K(t,x,u)u^i(x)dx$ transformed to

$\frac{1}{\alpha k}\sum_{k_{i-1=0}}^{k-\frac{1}{\alpha}}\sum_{k_{i-2}}^{k_{i-1}}\dots\sum_{k_{2=0}}^{k_3}\sum_{k_{1=0}}^{k_2}U_1(k_1)U_2(k_2-k_1)..U_{i-1}(k_{i-1}-k_i-\frac{1}{\alpha})*\lambda*$ $F\{K(t,x,u)\}$ $\forall\, i = 1,\dots,n$

In this part of the transform, k satisfies that $k\geq\frac{1}{\alpha}$,taking into consideration what is suitable for each function in terms of transformation.

Next we can characterize the new equation to find $U(k+\frac{\beta}{\alpha}),\ k=0,\dots,n$.

such that

$$U(k+\frac{\beta}{\alpha})=\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\beta+1)}[\,F(k)+\frac{1}{\alpha k}\sum_{k_{i-1=0}}^{k-\frac{1}{\alpha}}\sum_{k_{i-2}}^{k_{i-1}}\dots\sum_{k_{2=0}}^{k_3}\sum_{k_{1=0}}^{k_2}U_1(k_1)U_2(k_2-k_1)..U_{i-1}(k_{i-1}-k_i-\frac{1}{\alpha})]\ *\lambda*F\{K\,(t,x,u)]\qquad(3.2)$$

Now we have two cases:

First case When $\beta=\alpha=1$

To transform the initial condition of (3.1) we need to use the following relation at t=a

$$U(k_\circ)=\begin{cases}U(k_\circ)=\frac{1}{\alpha k_\circ}*\frac{du}{dt} & \text{If } \alpha k\in\mathbb{Z}^+\\ U(k_\circ)=0 & \text{If } \alpha k\notin\mathbb{Z}^+\quad \forall\, k_\circ=0,\dots,n\end{cases}$$

where $k_\circ=\frac{\beta}{\alpha}-1$,at t=0 .

It is clear that $k_\circ=0$ in this case, and by substituting $\beta=\alpha=1$ the value of $U(k+\frac{\beta}{\alpha})$ will be $U(k+1)$. Next substituting k values in the obtained equation $\forall\, k=0,\dots,n$. One can find the values of $U(k_i+1)\ \forall\, i=0,\dots,n$ which present the transformed series of $U(k_i+1)$,after this depending on the derivations of equation (1.6) in properties (2.2).Taking the inverse transform by using the following relation

$$u(t)=\sum_{k=0}^{\infty}U(k)(t-t_\circ)^{\alpha k}\ t_\circ=0\quad \alpha=1\ ,\ u(t)=\sum_{k=0}^{\infty}U(k)t^{\alpha k}$$

We get the semi analytic solution for equation (3.1) in series form.

Second case when β is fractional

In this case α must satisfies:

- $\alpha \le \beta - 1$.
- $\frac{\beta}{\alpha} \in \mathbb{Z}^+$

By the same way we can substitute values of β and α in $U(k+\frac{\beta}{\alpha}), k = 0, \ldots, n$. and apply the same steps to obtain the transformed initial condition . Then, we take k values $\forall\, k = 0, \ldots, n$, to find $U(k_i+\frac{\beta}{\alpha})\,\forall\, i = 0, \ldots, n$, the inverse transform is:

$$u(t) = \sum_{k=0}^{\infty} U(k)(t - t_\circ)^{\alpha k} \qquad t_\circ = 0 \quad , \alpha \text{ is fractional} \;, \quad u(t) = \sum_{k=0}^{\infty} U(k)t^{\alpha k}$$

to obtain the approximation solution for the original equation (3.1) in series form.

4. Solving system of Non Linear Volterra integro- differential equations of fractional order Using GDTM Technique,[**6**], [**7**], [**8**]:

In this section we modified the previous technique to solve a system of Non Linear Volterra integro- differential equations of fractional order .

Consider the system of NL-FVIDE

$$\sum_{j=1}^{n} D_c^{\beta} u_j(t) \; = f_j(t) + \lambda_j \int_0^t K_j\big(t, x, u_j\big) u_j^{(i)}(x)dx \quad (4.1)$$

With initial conditions $u_j(0)= a$, $0< \beta \le 1$, j=1,2,..,m ,i=1,2,..,n $m, n, \in \mathbb{Z}^+, \lambda_j \in \mathbb{R}$

$D_c^{\beta} u_j(t)$ denotes the caputo fractional derivative of order β for $u_j(t)$, $f_j(t)$ is continuous function with $f(t) \in C_{\mu}^{n}, t \in [a,b]$.To solve the system (4.1) by using GDTM, one can take the differential transform for both sides of system(4.1). According to GDTM properties in (2.2), the terms of equation (4.1)can be transformed as following

1- $D_c^{\beta} u_j(t)$ transformed to $\frac{\Gamma(\alpha k+\beta+1)}{\Gamma(\alpha k+1)} U_j(k+\frac{\beta}{\alpha})$

2- $f_j(t)$transformed to $F_j(k)$

3-

$\lambda \int_0^t K(t,x,u) u^{(i)}(x)dx$ transformed to $\frac{1}{\alpha k} \sum_{k_{i-1=0}}^{k-\frac{1}{\alpha}} \sum_{k_{i-2}}^{k_{i-1}} \ldots \sum_{k_{2=0}}^{k_3} \sum_{k_{1=0}}^{k_2} U_1(k_1) U_2(k_2 - k_1)..U_{i-1}(k_{i-1} - k_i - \frac{1}{\alpha})]\ \lambda\ F_j\{K(t,x,u)\}\ \forall\ i = 1, \ldots, n,$

and $\lambda_j \int_0^t K_j(t,x) u_j\ (x)dx$ transformed to $\frac{1}{\alpha k} U_j\left(k - \frac{1}{\alpha}\right)\ \lambda_j\ F\{K_j(t,x)\}$,

then:

$\lambda_j \int_0^t K_j(t,x,u_j\ ) u_j^i\ (x)dx$ transform to $\frac{1}{\alpha k} \sum_{j=1}^{m} U_j \sum_{k_{i-1=0}}^{k-\frac{1}{\alpha}} \sum_{k_{i-2}}^{k_{i-1}} \ldots \sum_{k_{2=0}}^{k_3} \sum_{k_{1=0}}^{k_2} U_1(k_1) U_2(k_2 - k_1)..U_{i-1}\left(k_{\,i-1} - k_i - \frac{1}{\alpha}\right]\ \lambda_j\ F_j\{K_j\left(t,x,u_j\ \right)\quad \forall\ i = 1, \ldots, n$

In this part of the transform k satisfies that $k \geq \frac{1}{\alpha}$,taking into consideration what is suitable for each function in terms of transformation. Next we can characterized the new equation to find

$U_j(k+\frac{\beta_j}{\alpha})$, $j = 1,2,..,m$, $k = 0, \ldots, n$. such that:

$U_j(k+\frac{\beta_j}{\alpha}) =$

$\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\beta_j+1)}[\,F(k) + \frac{1}{\alpha k}\sum_{j=1}^{m} U_j \sum_{k_{i-1}=0}^{k-\frac{1}{\alpha}} \sum_{k_{i-2}}^{k_{i-1}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} U_1(k_1)U_2(k_2 -$

$k)..U_{i-1}\left(k_{\,i-1} - k_i - \frac{1}{\alpha}\right] \lambda_j \;\; F_j\{K_j\left(t, x, u_j\;\right)\right]$ (4.2)

Now we have three cases to solve the cases of the derivative order types

First case When $\beta_j = \alpha \;\; = 1$

To transform the initial conditions we need to use the relation below

at t=a $U_j(k_\circ) = \begin{cases} \text{If } \alpha k \in \mathbb{Z}^+ \text{ then } U_j(k_\circ) = \frac{1}{\alpha k_\circ}\frac{du}{dt} \\ \text{If } \alpha k \notin \mathbb{Z}^+ \text{ then } U_j(k_\circ) = 0 \quad \forall\, k_\circ = 0, \dots, n \quad, \forall\, j = 1,2,..,m \end{cases}$

(4.3)

where $k_\circ = \frac{\beta_j}{\alpha} - 1$, a=0 . It is Clear that $k_\circ = 0$ in this case, and by substituting $\beta_j = \alpha \;\; = 1$ the value of $U_j(\,k+\frac{\beta_j}{\alpha})$ will be $U_j(k+1)$ $\forall$ j=1,2,..,m. Next substituting k values in the obtained equations $\forall\, k \;\; = 0, \dots, n$ .one can find the values of $U_j(\,k+1)$ $\forall\, k = 0, \dots, n$,which present the transformed series of $U_j(\,k+1)$, after this taking the inverse transform of equation (4.2) by using the relation

$u_j\,(t)=\sum_{k=1}^{\infty} U_j(k)(t - t_0)^{\alpha k}$ t_0=0, $\alpha = 1$, $u_j(t) = \sum_{k=1}^{\infty} U_j(k)t^k$

we get the semi analytic solution for the system of equations (4.1) in series frame.

Second case When β_j is fractional and $\beta_1 = \beta_2 = \beta_3 = \dots = \beta_j$, $\forall\, j = 1, 2, .., m$

α must satisfies :

- $\alpha \leq \beta_j - 1$.
- $\frac{\beta_j}{\alpha} \in \mathbb{Z}^+$

using the same way as in (4.3), we can substitute values of β_j and α in $U_j(k+\frac{\beta_j}{\alpha}), k = 0, \ldots, n$, and apply the same steps in (3.a) to obtain the transformed initial conditions $U_j(k_\circ)$.Then, we take k values $\forall k = 0, \ldots, n$ to find $U_j(k_i+\frac{\beta_j}{\alpha}), \forall j = 1,2, \ldots, m$, $\forall i = 0, \ldots, n$. ,after this, we take the inverse transform of equation(4.2)

$u_j(t)=\sum_{k=1}^{\infty} U_j(k)(t-t_0)^{\alpha k}$ $\quad t_0=0$, α is fractional , $u_j(t) = \sum_{k=0}^{\infty} U_j(k)t^{\alpha k}$

To obtain the approximation solution for the original system of equations (4.1) in series form.

Third case:-When β_j is fractional and $\beta_1 \neq \beta_2 \neq \beta_3 \neq \cdots \neq \beta_j$,$\forall$ j = 1, 2, . . , m

α must satisfies:

- $\alpha \leq \beta_j - 1$.
- $\frac{\beta_j}{\alpha} \in \mathbb{Z}^+$

By using the same way in (4.3), we can substitute values of β_j and α in $U_j(k+\frac{\beta_j}{\alpha})$, $k = 0, \ldots, n$,and apply the same steps to obtain the transformed initial conditions $U_j(k_\circ)$.We may find out that the numbers of the transformed initial conditions $U_j(k_\circ)$are different for equation to other according to value of β_j for each equation.

Then next take k values $\forall k = 0, \ldots, n$, to find $U_j(k_i+\frac{\beta_j}{\alpha}), \forall j = 1,2, \ldots, m$,$\forall i = 0, \ldots, n$.

after this, we take the inverse transform of equation (4.2)

$u_j(t)=\sum_{k=1}^{\infty} U_j(k)(t-t_0)^{\alpha k}$ $t_0=0$ α is fractional , $u_j(t)=\sum_{k=0}^{\infty} U_j(k)t^{\alpha k}$

in order to obtain the approximation solution for the original equation (4.1) in series form.

5. Numerical example

Consider we have system of NL-FVIDE :

$$D_c^{\beta} u_1(t) = -2-3t-2\sin t+3\cos t+\int_0^t[u_1^2(x)+u_2^2(x)]dx$$

$$D_c^{\beta} u_2(t) = -2+\sin t+2\cos t+\frac{1}{2}\sin 2t+\int_0^t[u_1^2(x)-u_2^2(x)]dx \qquad (5.1)$$

With initial conditions $u_1(0)=1$, $u_2(0)=2$

the exact solution is given in [1] as: $u_1(t)=1+\sin t$, $u_2(t)=1+\cos t$

To solve the system (5.1) using GDTM Technique and the properties in (2.2), we get the transformed equation below: For the first equation

$$U_1\left(k+\frac{\beta}{\alpha}\right)=\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\beta+1)}\left[-2\delta(k)-3\delta\left(k-\frac{1}{\alpha}\right)-\frac{2}{k!}\sin\left(\frac{\pi k}{2}\right)+\frac{3}{k!}\cos\left(\frac{\pi k}{2}\right)\right.$$
$$\left.+\frac{1}{\alpha k}\sum_{k_1=0}^{k}U_1(k_1)\,U^{1\left(k-k_1-\frac{1}{\alpha}\right)}+\frac{1}{\alpha k}\sum_{k_1=0}^{k}U_2(k)U_2(k-k_1-\frac{1}{\alpha})\right] \quad (5.1.a)$$

By the same way we can transform the second equation:

$$U_2\left(k+\frac{\beta}{\alpha}\right)=\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\beta+1)}\left[2\delta(k)+\frac{1}{k!}\sin\left(\frac{\pi k}{2}\right)+\frac{1}{\alpha k}\sum_{k_1=0}^{k}U_1(k_1)\,U_1(k-k_1-\frac{1}{\alpha})\right.$$
$$\left.-\frac{1}{\alpha k}\sum_{k_1=0}^{k}U_2(k)U_2(k-k_1-\frac{1}{\alpha})\right] \quad (5.1.b)$$

First case:

To obtain the first solution: put $\alpha = \beta = 1$ which means $k_\circ=0$ then U_1 (0) = 1 , U_2 (0) = 2

Then substituting $\alpha$and β values in equations (5.1.a), (5.1.b) we get:

$$U_1\ (k+1)=\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+2)}[-2\delta(k)-3\delta(k-1)-\frac{2}{k!}\sin\left(\frac{\pi k}{2}\right)+\frac{3}{k!}\cos\left(\frac{\pi k}{2}\right)+\frac{1}{k}\sum_{k_1=0}^{k}U_1(k_1)U_1(k-k_1-1)+\frac{1}{k}\sum U_2(k_1)U_2(k-k_1-1)]$$

$$U_2\ (k+1)=\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+2)}[-2\delta(k)+\frac{1}{k!}\sin\left(\frac{\pi k}{2}\right)+\frac{2^{k-1}}{k!}\sin\left(\frac{\pi k}{2}\right)-\frac{1}{k}\sum_{k_1=0}^{k}U_1(k_1)U_1(k-k_1-1)-\frac{1}{k}\sum_{k_1=0}^{k}U_2(k_1)U_2(k-k_1-1)] \quad (5.1.c)$$

Substituting the values k in system equations (5.1. c), $\forall\ k = 0,1,2,\ldots$

For k=0 then $U_1(1) = 1$, for $k = 1$ then $U_1(2) = 0$

For k=0 then $U_2(1) = 0$, for $k = 1$ then $U_2(2) = -0.5000000000$

Continue this process $\forall k \geq 2$ one can find $U_1(3),\ldots$..and $U_2(3),\ldots$.

Now to get semi analytic solution for system (5.1) formed in a series form applying the inverse transform of system (5.1.c):

u_1 (t)=$\sum_{k=1}^{\infty}U_1(k)(t-t_0)^{\alpha k}$ t_0=0, $\alpha = 1$, $u_1(t) = \sum_{k_1=0}^{\infty}U(k)t^k$

$$u_1(t) = U_1(0)t^0 + U_1(1)t + U_1(2)t^2 + U_1(3)t^3 + U_1(4)t^4 + U_1(5)t^5 + \cdots + U_1(n)t^n + \cdots$$

$$u_1(t) = 1 + t + (0)*t^2 + (\tfrac{-1}{3!})t^7 + (0)t^4 + \left(\tfrac{1}{5!}\right)t^5 + ..$$

$$u_1(t) = 1 + t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \frac{1}{9!}t^9 + \cdots$$

Then $u_1(t) = 1 + \mathrm{sint}$ and this is exact solution.

By the way we can find $u_2(t)$

$u_2(t)=\sum_{k=1}^{\infty} U_2(k)(t-t_0)^{\alpha k}$ $t_0=0$, $\alpha = 1$, $u_2(t) = \sum_{k_1=0}^{\infty} U_2(K)t^k$

$$u_2(t) = U^{2(0)t^0} + U_2(1)t + U_2(2)t^2 + U_2(3)t^3 + U_2(4)t^4 + U_2(5)t^5 + \cdots + U_2(n)t^n + \cdots$$

$$u_2(t) = 1 + (0) + \left(-\frac{1}{2!}\right)t^2 + (0) * t^3 + \left(\frac{-1}{4!}\right)t^4 + (0) * t^5 + ..$$

$$u_2(t) = 1 - \frac{1}{2!}t^2 - \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + ..$$

Then $u_2(t) = 1 + \mathrm{cost}$ and this is exact solution.

Second case:

For this case one can select the value of $\beta_j\ \forall\ j = 1,2$ as: $\beta_1 = \beta_2 = 0.5$, $\alpha = 0.5$

which means $k_\circ = 0$ then $U_1(0) = 1$, $U_2(0) = 2$

And the equations will be: $U_1(k+1) = \frac{\Gamma\left(\frac{k}{2}+1\right)}{\Gamma\left(\frac{k}{2}+\frac{1}{2}+1\right)}\left[-\delta(k) - 3\delta(k-2) - \frac{2}{k!}\sin\left(\frac{\pi k}{2}\right) + \frac{3}{k!}\cos\left(\frac{\pi k}{2}\right) + \frac{2}{k}\sum_{k_1=0}^{k} U_1(k)U_1(k-k_1-2) + \frac{2}{k}\sum_{k_1=0}^{k} U_2(k)U_2(k-k_1-2)\right]$

$$U_2(k+1) = \frac{\Gamma\left(\frac{k}{2}+1\right)}{\Gamma\left(\frac{k}{2}+\frac{1}{2}+1\right)}\left[-2\delta(k) + \frac{1}{K!}\sin\left(\frac{\pi k}{2}\right) + \frac{2}{k}\sum_{k_1=0}^{k} U_1(k)U_1(k-k_1-2) - \frac{2}{k}\sum_{k_1=0}^{k} U_2(k)U_2(k-k_1-2)\right] \quad (5.1.d)$$

For k=0 then $U_1(1) = 1.1283792$

For k=1 then $U_1(2) = 1.7724539$, $U_2(2) = 1.7724539$

Continue this process $\forall k \geq 2$ one can find $U_1(3), \ldots$ and $U_2(3), \ldots\ldots$

gain applying the inverse transform of system (5.1.d) to get the approximate solution for system (5.1) formed in a series form: $u_1(t)=\sum_{k=1}^{\infty} U_1(k)(t-t_0)^{\alpha k}$ $t_0=0$, $\alpha = \frac{1}{2}$, $u_1(t)=\sum_{k=0}^{\infty} U_1(k)t^{\frac{1}{2}k}$

$$u_1(t) = U_1(0)t^0 + U_1(1)t^{\frac{1}{2}} + U_1(2)t^2 + U_1(3)t^{\frac{3}{2}} + \cdots + U_1(n)t^{\frac{n}{2}} + \cdots$$

$$u_1(t)= 1+(1.1283792)\, t^{\frac{1}{2}} + (1.7724539)t^2+(-1.6874848)t^{\frac{3}{2}} + \cdots + U_1(n)t^{\frac{n}{2}} + \cdots$$

By the way we can find $u_2(t)$

$u_2(t)=\sum_{k=1}^{\infty} U_2(k)(t-t_0)^{\alpha k}$ $t_0=0, \alpha = \frac{1}{2}$, $u_2(t)=\sum_{k=0}^{\infty} U_2(K)t^{\frac{1}{2}k}$

$$u_2(t) = U_2(0)t^0 + U_2(1)t^{\frac{1}{2}} + U_2(2)t^2 + U_2(3)t^{\frac{3}{2}} + \cdots\ U_2(n)t^{\frac{n}{2}} + \cdots$$

$$u_2(t) = 2 + 0 + (1.7724539\,)t^2 + (0.9453999\,)t^{\frac{3}{2}} + \cdots\ U_2(n)t^{\frac{n}{2}} + \cdots$$

Finding the arbitrary value of t for any fractional divertive order β and calculate it:

For $\beta = 0.8$ then $\alpha = 0.2$. By substituting β and α values in equation (5.1.a) and (5.1.b) we get

$$U_1(k+4)=\frac{\Gamma\left(\frac{k}{5}+1\right)}{\Gamma\left(\frac{k}{5}+\frac{4}{5}+1\right)}\left[-2\delta(k)-3\delta(k-5)-\frac{2}{k!}\sin\left(\frac{\pi k}{2}\right)+\frac{3}{k!}\cos\left(\frac{\pi k}{2}\right)\right.$$
$$\left.+\frac{5}{k}\sum_{k_1=0}^{k}U_1(k_1)\,U_1(k-k_1-5)+\frac{5}{k}\sum_{k_1=0}^{k}U_2(k)U_2(k-k_1-5)\right]$$

$$U_2(k+4)=\frac{\Gamma\left(\frac{k}{5}+1\right)}{\Gamma\left(\frac{k}{5}+\frac{4}{5}+1\right)}\left[2\delta(k)+\frac{1}{k!}\sin\left(\frac{\pi k}{2}\right)+\frac{5}{k}\sum_{k_1=0}^{k}U_1(k_1)\,U_1(k-k_1-5)\right.$$
$$\left.-\frac{5}{k}\sum_{k_1=0}^{k}U_2(k)U_2(k-k_1-5)\right]\quad(5.1.g)$$

For example take k=0 then: $U_1(4)=1.0736713$

and by the same way we can find U_1 (5), U_1 (6),..., U_1 (10) $\forall k\geq 1$,

By the way we can find $u_2(t)$ for k=0 then $U_2(4)=0$

and by the same way we can find U_2 (5), U_2 (6),..., U_2 (10) $\forall k\geq 1$,

applying the inverse transform of system (5.1.g) to get the approximate solution for system (5.1) formed in a series form: u_1 (t)=$\sum_{k=1}^{\infty}U_1(k)(t-t_0)^{\alpha k}$ t_0=0, $\alpha=\frac{1}{5}$

$$u_1(t)=U_1(0)t^0+U_1(1)t^{\frac{1}{5}}+U_1(2)t^1+U_1(3)t^{\frac{3}{5}}+\cdots+\ldots U_1(n)\,t^{\frac{n}{5}}+\cdots$$

$u_1(t)$= 1+ (1.1929681) $t^{\frac{9}{5}}$+(7.3453499)t+ ….$U_1(n)\,t^{\frac{n}{5}}+\cdots$

u_2 (t)=$\sum_{k=1}^{\infty}U_2(k)(t-t_0)^{\alpha k}$ t_0=0, $\alpha=\frac{1}{5}$

$$u_2(t)=U_2(0)t^0+U_2(1)t^{\frac{1}{5}}+U_2(2)t^1+U_2(3)t^{\frac{3}{5}}+U_2(4)t^{\frac{4}{5}}+\cdots+\ldots U_2(n)\,t^{\frac{n}{5}}+\cdots$$

$$u_2(t)=(1.8363375)t+\ldots+U_2(n)\,t^{\frac{n}{5}}+\cdots$$

For t=0.6 , $\beta=0.8$,$\alpha=0.2$,we can obtain $u_1(t)=13.80561722$, $u_2(t)=3.64345194$

The illustrated value of u(t) colored as red in table (2) below ,one can find the other entries values of table (2)by the same way.

Third case: Putting $D_c^{\beta_1}u_1(t)=-2-3t-2\sin t+3\cos t+\int_0^t[u_1^{\,2}(x)+u_2^{\,2}(x)]dx$

$$D_c^{\beta_2}u_2(t)=-2+\sin t+2\cos t+\frac{1}{2}\sin 2t+\int_0^t[u_1^{\,2}(t)-u_2^{\,2}(t)]dt$$

With initial conditions $u_1(0)=1$, $u_2(0)=2$

the exact solution is given in [1] as : $u_1(t)=1+\sin t$, $u_2(t)=1+\cos t$

$$U_1\left(k+\frac{\beta_1}{\alpha}\right)=\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\beta_1+1)}[-2\delta(k)-3\delta\left(k-\frac{1}{\alpha}\right)-\frac{2}{k!}\sin\left(\frac{\pi k}{2}\right)+\frac{3}{k!}\cos\left(\frac{\pi k}{2}\right)$$
$$+\frac{1}{\alpha k}\sum_{k_1=0}^{k}U_1(k_1)U_1\left(k-k_1-\frac{1}{\alpha}\right)+\frac{1}{\alpha k}\sum_{k_1=0}^{k}U_2(k_1)U_2\left(k-k_1-\frac{1}{\alpha}\right)]$$

$$U_2\left(k+\frac{\beta_2}{\alpha}\right)=\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\beta_2+1)}[-2\delta(k)+\frac{1}{k!}\sin\left(\frac{\pi k}{2}\right)+\frac{2}{k!}\cos\left(\frac{\pi k}{2}\right)+\frac{2^{k-1}}{k!}\sin\left(\frac{\pi k}{2}\right)+$$
$$+\frac{1}{\alpha k}\sum_{k_1=0}^{k}U_1(k_1)U_1\left(k-k_1-\frac{1}{\alpha}\right)-\frac{1}{\alpha k}\sum_{k_1=0}^{k}U_2(k_1)U_2\left(k-k_1-\frac{1}{\alpha}\right)]\qquad(5.1.f)$$

put:- $\beta_1=0.5=\frac{1}{2}$, $\beta_2=0.75=\frac{3}{4}$, $\alpha=0.25=\frac{1}{4}$

To find transform initial conditions : $k_1=\frac{\beta_1}{\alpha}-1=1$ which means $k_1=0,1$

$k_2 = \frac{\beta_2}{\alpha} - 1 \quad = 2$ which means $k_2 = 0,1,2$

Then : $U_1(0) = 1$, $U_1(1) = 0$ and $U_2(0) = 2$, $U_2(1) = U_2(2) = 0$

Then substituting α and β values in system (5.1.f) $\forall\, k = 0,1,2, \ldots.$

$$U_1(k+2) = \frac{\Gamma\left(\frac{k}{4}+1\right)}{\Gamma\left(\frac{k}{4}+\frac{1}{2}+1\right)}\Big[-2\delta(k) - 3\delta(k-4) - \frac{2}{k!}\sin\left(\frac{\pi k}{2}\right) + \frac{3}{k!}\cos\left(\frac{\pi k}{2}\right) + \frac{2}{k}\sum_{k_1=0}^{k} U_1(k_1)U_1(k-k_1-4) + \frac{2}{k}\sum_{k_1=0}^{k} U_2(k_1)U_2(k-k_1-4)\Big]$$

$$U_2(k+3) = \frac{\Gamma\left(\frac{k}{4}+1\right)}{\Gamma\left(\frac{k}{4}+\frac{3}{4}+1\right)}\Big[-2\delta(k) + \frac{1}{k!}\sin\left(\frac{\pi k}{2}\right) + \frac{2}{k!}\cos\left(\frac{\pi k}{2}\right) + \frac{2^{k-1}}{k!}\sin\left(\frac{\pi k}{2}\right) + \frac{4}{k}\sum_{k_1=0}^{k} U_1(k_1)U_1(k-k_1-4) + \frac{4}{K}\sum_{k_1=0}^{k} U_2(k_1)U_2(k-k_1-4)\Big] \qquad (5.1.h)$$

Substituting k values in system (5.1.h) $\forall\, k = 0,1,2, \ldots.$

For k=0 then $U_1(2) = 1.1283792$ $\quad U_2(3) = 0$

For $k = 1$ then $U_1(3) = 5.9173502$ $\quad U_2(4) = 1.8128050$

Continue this process $\forall\, k \geq 2$ we can find $U_1(4), \ldots$ and $U_2(5), \ldots$

again applying the inverse transform of system (5.1.h) to get the approximate solution for system (5.1) formed in a series form:

$u_1(t) = \sum_{k=1}^{\infty} U_1(k)(t-t_0)^{\alpha k}$ $\quad t_0=0,\ \alpha = \frac{1}{4}$ $\quad u(t) = \sum_{k=0}^{\infty} U(k)t^{\frac{1}{4}k}$

$$u_1(t) = U_1(0)t^0 + U_1(1)t^{\frac{1}{4}} + U_1(2)t^{\frac{1}{2}} + U_1(3)t^{\frac{3}{4}} + \cdots \ldots \ldots$$

$$u_1(t) = 1 + 0 + \ldots + 0 + (1.1283792)t^{\frac{1}{2}} + (5.9173502)t^{\frac{3}{4}} + \ldots$$

$u_1(t) = 1 + (1.1283792)t^{\frac{1}{2}} + (5.9173502)t^{\frac{3}{4}} + \cdots$

By the way we can find $u_2(t)$, $u_2(t)=\sum_{k=1}^{\infty} U_2(k)(t-t_0)^{\alpha k}$ $t_0=0, \alpha = \frac{1}{4}$, $u_2(t)=\sum_{k=0}^{\infty} U_2(k)t^{\frac{1}{4}k}$

$u_2(t) = U_2(0)t^0 + U_2(1)t^{\frac{1}{4}} + U_2(2)t^{\frac{1}{2}} + U_2(3)t^{\frac{3}{4}} + \cdots \ldots \ldots$

$u_2(t)= 2 + 0 + 0 + 0 + (1.8128050)t + (-0.7821929)t^{\frac{5}{4}} + \ldots$

$u_2(t)= 2 + (1.8128050)\, t + (-0.7821929)t^{\frac{5}{4}} + \ldots + \cdots + \ldots U_2(n)\, t^{\frac{n}{5}} + \cdots$

4. Conclusions

The differential transformation method has been applied to obtain the numerical solution of nonlinear systems of integro-differential equations of fractional order. The present method has been applied in a direct way without using linearization, discretization , or perturbation. Comparison of the numerical results with the existing technique [15] shows that the present method is more accurate. The proposed method is able and related to a wide class of linear and nonlinear problems in the theory of fractional calculus.

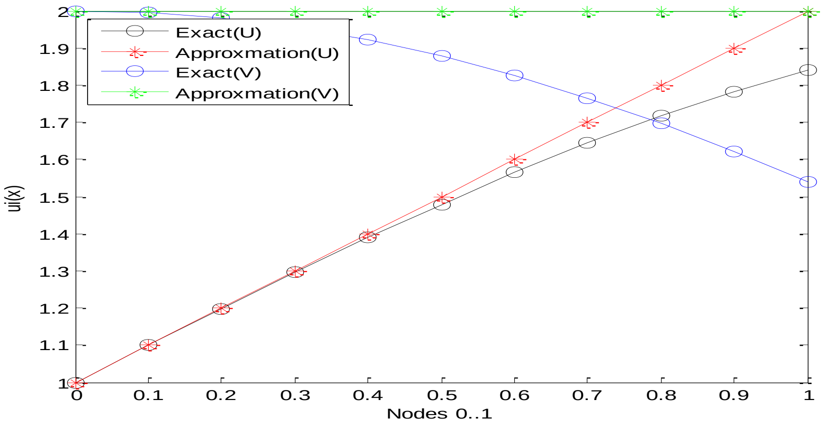

Figure (1): Comparison between the exact solution and approximate solution of the example using GDTM when N=1 ,β=1

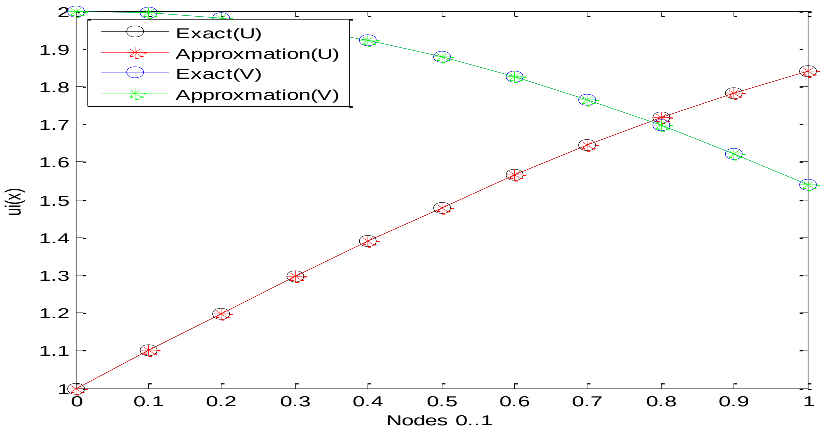

Figure (2): Comparison between the exact solution and approximate solution of the example using GDTM when N=10 ,β=1

Table (1)shows the Results of system of the NonLinaer Volterra integro- differential equations of fractional order solved using GDTM after considering different values for N, $\boldsymbol{\beta = 1}$

Values of t	$\alpha = \beta = 1$ N=1		$\alpha = \beta = 1$ N=10	
	u_1	u_2	u_1	u_2
0.0	1.00000000	2.00000000	1.00000000	2.00000000
0.1	1.10000000	1.99500417	1.09983342	1.99500417
0.2	1.20000000	1.98006658	1.19866933	1 .98006658
0.3	1.30000000	1.95533649	1.29552021	1.95533649
0.4	1.40000000	1.92106099	1.38941834	1.92106099
0.5	1.50000000	1.87758256	1.47942554	1.87758256
0.6	1.60000000	1.82533561	1.56464247	1 .82533561
0.7	1.70000000	1.76484219	1.64421769	1.76484219
0.8	1.80000000	1.69670671	1.71735609	1 .69670671
0.9	1.90000000	1.62160997	1.78332692	1 .62160997
1.0	2.00000000	1.54030231	1.84147101	1.54030230
s =	0.0486	0.1822	3.5454e-009	2.8401e-010
ans =	0.0046	0.0505	6.2331e-017	4.2515e-019

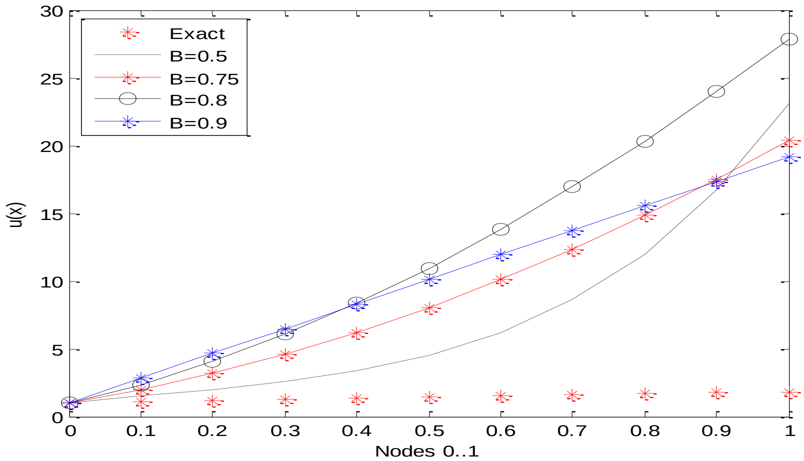

Figure (3):Comparison between the approximate solutions for u_1 in the example using GDTM when N=10 for different values for the fractional order divertive β

Values of t	$\beta = 0.5$		$\beta = 0.8$		$\beta = 0.9$	
	u_1	u_2	u_1	u_2	u_1	u_2
0.0	1.0000000	2.000000	1.0000000	2.000000	1.0000000	2.0000000
0.1	1.54274391	2.19254637	2.3503841	2.14536899	2.84332868	2.19027015
0.2	1.98103556	2.39552639	4.05107797	2.33118164	4.66912575	2.38054031
0.3	2.55553955	2.63378990	6.06170287	2.57085483	6.48912976	2.57081046
0.4	3.36536338	2.95121054	8.36424217	2.86822943	8.30553638	2.76108062
0.5	4.53147019	3.40729395	10.94798810	3.22533361	10.11934737	2.95135077
0.6	6.21257892	4.07804382	13.80561722	3.64345194	11.93113677	3.14162092
0.7	8.61497071	5.05753887	16.93170779	4.12348196	13.74127591	3.33189108
0.8	12.00092396	6.45977772	20.32203392	4.66609119	15.55002419	3.52216123
0.9	16.69670699	8.42065855		5.27179825	17.35757273	3.71243139

			3.97318012			
1.0	23.10039953	11.1000394	27.88231024	5.94101920	19.16406799	3.90270154

Table (2) shows Results of system of the NonLinaer Volterra integro- differential equations of fractional order solved using GDTM after considering different values for the fractional order divertive β

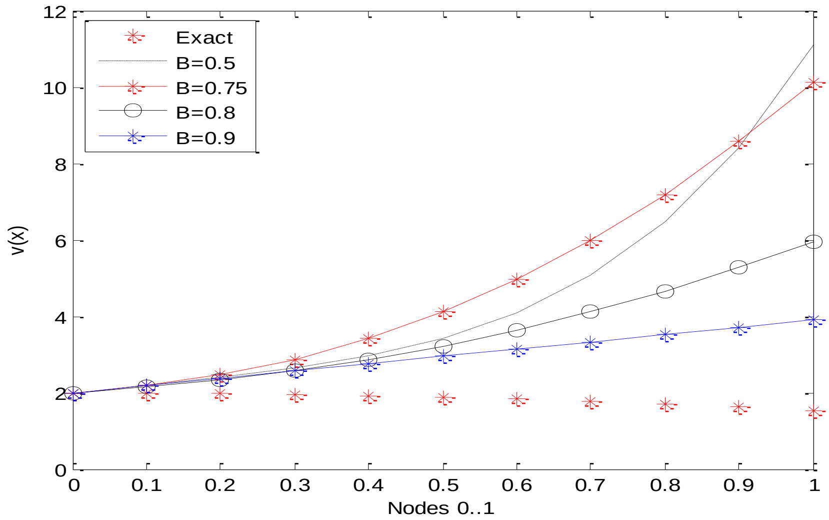

Figure (4): Comparison between the approximate solutions for u_2 in the example using GDTM when N=10 for different values for the fractional order divertive β

Table (3)shows Results of system of the NonLinaer Volterra integro- differential equations of fractional order solved using GDTM after considering mixed values for the fractional order divertive β

Values of t	$\beta_1 = 0.5$, $\beta_2 = 0.75$, $\alpha = 0.25$ N=10	
	u_1	u_2
0.0	1.00000000	2.00000000
0.1	3.35314506	2.33514642
0.2	6.25155942	3.09923531
0.3	10.17281514	4.37540089
0.4	15.25188735	6.22716788
0.5	21.58741068	8.70843062
0.6	29.26186330	11.86686495
0.7	38.34800947	15.74568268
0.8	48.91187331	20.38468424
0.9	61.01440658	25.82095121
1.0	74.71254269	32.08933156

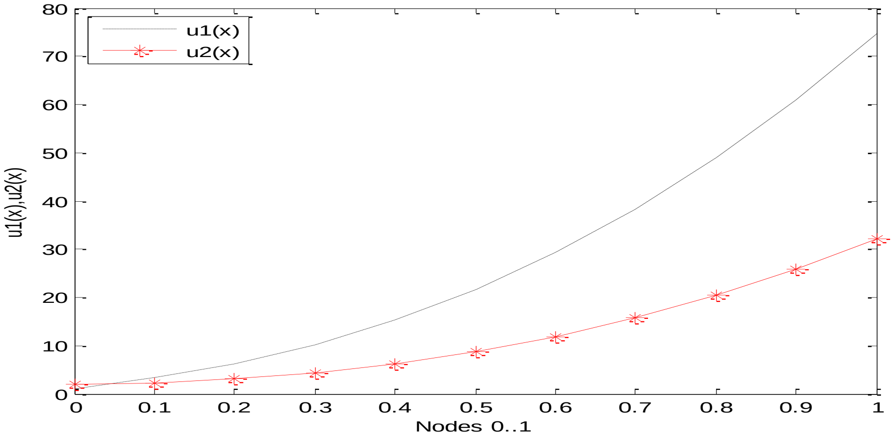

Figure (5):Comparison of the approximation solutions for u_1 and u_2 in the example using GDTM when N=10 for mixed values for the fractional order divertive β

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