ON δ-I-proper function

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Abstract

In this paper,we introduce a new class of function called δ-I-proper function and explain some of propositions,theorems and some equivalent statements of this function .

Key words and phrases: δ-I-cluster point,R-I-open set ,δ-I-open set , δ-I-continuous function,δ-I-closed function,δ-I-proper function .

1- Introduction:

Let A be a subset of a topological space (X,T).The **δ-interior** of a subset A of X is the union of all regular open (R-open) sets of contained in A and is denoted by $Int_\delta(A)$ [7] . The subset A is called **δ-open** if $A=Int_\delta(A)$ (i.e. a set A is **δ-open** if it is the union of regular open sets)[7] .The complement of a **δ-open** set is called **δ-closed** . Alternatively , a set $A\subseteq(X,T)$ is called **δ-closed** if $A=Cl_\delta(A)$, where $Cl_\delta(A)=\{x\in X:Int(Cl(U))\cap A\neq\emptyset,\ U\in T$ and $x\in U\}$[1].The family of all **δ-open** sets forms a topology on X and denoted by T_δ .Since the intersection of two regular open sets is regular open , the collection of all regular open sets forms a base for a coarser topology T_S than the original one T . A basic facts are that $T_S= T_\delta$ and every clopen set is regular open set [7] . In a topological space (X,T) , let *I* an ideal of subsets of X .An ideal is defined as a non-empty collection *I* of subsets of X satisfying two conditions : 1) If $A\in I$ and $B\subseteq A$, then $B\in I$; 2) If $A\in I$ and $B\in I$, then $A\cup B\in I$ .A topological (X,T) with an ideal I on X is called ideal topological space and denoted by (X,T,I) . For a subset $A\subseteq X$, $A^*=\{x\in X:U\cap A\notin I\ ,\ U\in T$ and $x\in U\}$ is called the local function of A with respect to I and T [3] .For each ideal topological space (X,T,I) , there exists a topology T^* finer than T , generated by $\beta(I,T)=\{U\backslash I:U\in T$ and $I\in I\}$, but in general is not always a topology [2].Additionally , $Cl^*(A)=A\cup A^*$ defines a Kuratowiski closure for T^* and $T\subset T^*$.

1-1 Definition[9] : A subset A of an ideal topological space (X,T,I) is said to be an R-I-open set if $Int(Cl^*(A))=A$ .We call a subset A of X R-I-closed if its complement is R-I-open .

1-2 Definition[9] : Let (X,T,I) be an ideal topological space , S a subset of X and x is a point of X .

1- x is called a δ-I-cluster point of S if $S\cap Int(Cl^*(U))\neq\emptyset$ for each open neighborhood x ;

2- The family of all δ-I-cluster points of S is called the δ-I-closure of S and is denoted by $[S]_{\delta\text{-}I}$ and

3- A subset S is said to be δ-I-closed if $[S]_{\delta\text{-}I}=S$. The complement of a δ-I-closed set of X is said to be ,δ-I-open .

4- $[S]_{\delta\text{-}I}$ is δ-I-closed .

1-3 Definition :Let A be a subset of an ideal topological space (X,T,I) .A point $x\in A$ is called δ-I-interior point of A if there exists a R-I-open set U in X such that $x\in U\subseteq A$. The set of a δ-I-interior points of A is called δ-I-interior set and its denoted by $Int_{\delta\text{-}I}(A)$.

1-4 Proposition : Let (X,T,I) be an ideal topological space ,$A\subseteq X$ then A is a δ-I-open if and only if $A=Int_{\delta\text{-}I}(A)$.

Proof : →) Let A be a δ-I-open

(1) $Int_{\delta\text{-}I}(A)\subseteq A$ (by Def)

(2) Let $x\in A$, if $x\notin Int_{\delta\text{-}I}(A)\Longrightarrow$for each R-I-open U and $x\in U$ then $U\cap A^c\neq\emptyset \Longrightarrow x\in[A^c]_{\delta\text{-}I}\Longrightarrow x\in A^c$ C! $(A^c=[A^c]_{\delta\text{-}I})$

Hence by (1) and (2) $A=Int_{\delta\text{-}I}(A)$.

($\Leftarrow$ Let $A= Int_{\delta\text{-}I}(A)$

(1) Let $x\in([A^c]_{\delta\text{-}I})^c \Leftrightarrow x\notin[A^c]_{\delta\text{-}I} \Leftrightarrow$ there exist a R-I-open U and $x\in U$ such that $U\cap A^c=\emptyset \Leftrightarrow x\in U\subseteq A \Leftrightarrow x\in Int_{\delta\text{-}I}(A) \Leftrightarrow x\in A$.Hence $A= ([A^c]_{\delta\text{-}I})^c$ and A is a δ-I-open

1-5 Theorem[9]: Let (X,T,I) be an ideal topological space and $T_{\delta\text{-}I}=\{A\subseteq X: A$ is a δ-I-open set of $(X,T,I)\}$ .Then $T_{\delta\text{-}I}$ is a topological space s.t $T_s\subseteq T_{\delta\text{-}I}\subseteq T$.

1-6 Proposition : Let (X,T,I) be an ideal topological space ,$A\subseteq X$ then :

i) If A is clopen set,then it is δ-I-clopen.

ii) If A is δ-I-closed set,then it is closed.

iii) If $\{ A_\lambda \mid \lambda\in\Lambda \}$ is a family of δ-I-closed(i.e. $[A]_{\delta\text{-}I} =A$) , then $\bigcap_{\lambda\in\Lambda} A_\lambda$ is δ-I-closed .

v) If A is R-I-open set,then it is open.

Proof :

i) Let A be is clopen $\Rightarrow A^c$ clopen $\Rightarrow A= Int(Cl(A))$ and $A^c = Int(Cl(A^c))$.

For each $x\in A \Rightarrow x\in Int(A)\subseteq Int(Cl^*(A))\subseteq Int(Cl(A))\subseteq A \Rightarrow$ A is **δ-I-open** .

For each $x\in A^c \Rightarrow x\in Int(A^c)\subseteq Int(Cl^*(A^c))\subseteq Int(Cl(A^c))\subseteq A^c \Rightarrow A^c$ is δ-I-open$\Rightarrow$ A is **δ-I-closed** .Hence A is δ-I-clopen.

ii) Clear .

iii) Suppose that$\{ A_\lambda| \lambda\in\Lambda\}$ is a family of **δ-I-closed**,then $\{ A^c_\lambda \mid \lambda\in\Lambda \}$ is a family of **δ-I-open** .But $T_{\delta\text{-}I}$ is topology then $\cup_{\lambda\in\Lambda} A^c_\lambda$ is **δ-I-open** and hence $\cap_{\lambda\in\Lambda} A_\lambda$ is **δ-I-closed** .

v) Let A= A= Int(Cl*(A))

1- $\text{Int}(A) \subseteq A$

2- Let $x \in A$,if $x \notin \text{Int}(A)$, since $\text{Int}(\text{Cl}^*(A)) \subseteq A \Longrightarrow \text{Int}(\text{Int}(\text{Cl}^*(A))) \subseteq \text{Int}(A) \Longrightarrow \text{Int}(\text{Cl}^*(A)) \subseteq \text{Int}(A) \Longrightarrow x \notin \text{Int}(\text{Cl}^*(A)) \Longrightarrow x \notin A$ C! $\Longrightarrow x \in \text{Int}(A)$. Hence by (1) and (2) A is open set .

1-7 Lemma: Let (X,T,I) be an ideal topological space and A , Y subsets of X such that $A \subset Y$.Then A is **R-I-open** in (Y,T_Y,I_Y) if and only if

$A= \text{Int}(\text{Cl}^*(A)) \cap Y$.

1-8 Proposition : Let (X,T,I) be an ideal topological space A , Y subsets of X such that $A \subset Y$ .Then if A is **δ-I-closed** in (X,T,I),then A is **δ-I-closed** in (Y,T_Y,I_Y).

Proof : Let A be **δ-I-closed** in (X,T,I),then A^c is **δ-I-open** .

For each $x \in A^c$,there exists **R-I-open** U in X such that $x \in U \subseteq A^c \Longrightarrow U \cap A = \varnothing \Longrightarrow U \cap (A \cap Y) = \varnothing \Longrightarrow (U \cap Y) \cap A = \varnothing \Longrightarrow (U \cap Y) \subseteq A^c$ .By Lemma (1-7),$U \cap Y$ is **R-I-open** in Y and A^c is **δ-I-open** in Y .Hence A is **δ-I-closed** in (Y,T_Y,I_Y).

1-9 Definition : Let (X,T,I) be an ideal topological space , $A \subseteq X$. A **R-I-neighborhood (R-I-nbd)** of A is any subset of X which contains a **R-I-open** set containing A. we denote the collection of all **R-I-neighbor-hoods** of x belong to y $N_{R\text{-}I}(x)$.

1-10 Definition : Let $(\chi_d)_{d \in D}$ be a **net** in a space X , $x \in X$. Then :

(i) $(\chi_d)_{d \in D}$ is **δ-I-converges** to x [written $\chi_d \xrightarrow{\delta-I} x$] , if $(\chi_d)_{d \in D}$ is **eventually** in every **R-I-nbd** of x . The point x is called a **δ-I-limit** point of $(\chi_d)_{d \in D}$.

(ii) $(\chi_d)_{d \in D}$ is said to have x as a **δ-I-cluster** point [written $\chi_d \propto_{\delta-I} x$] if $(\chi_d)_{d \in D}$ is **frequently** in every **R-I-nbd** of x .

1-11 Remarks : Let (X,T,I) be an ideal topological space, then :

(i) If (χ_d) is a **net** in X , $x \in X$ such that $\chi_d \to x$ then $\chi_d \xrightarrow{\delta-I} x$.

(ii) If (χ_d) is a **net** in X , $x \in X$ such that $\chi_d \propto x$ then $\chi_d \propto_{\delta-I} x$.

(iii) If (χ_d) is a **net** in X , $x \in X$. Then $\chi_d \xrightarrow{\delta-I} x$ in (X, T, I) if and only if $\chi_d \to x$. in $(X, T_{\delta-I}, I)$, and $\chi_d \propto_{\delta-I} x$ in (X, T, I) if and only if $\chi_d \propto x$ in $(X, T_{\delta-I}, I)$.

1-12 Theorem [6] : A space X is **compact** if and only if every **net** in X has a **cluster** point in X .

1-13 Definition : An ideal topological space (X,T,I) is called **δ-I-compact** if every **δ-I-open** cover of X has a finite subcover .

1-14 Proposition : Let (X,T,I) be an ideal topological space and $A \subseteq X$, $x \in X$. Then $x \in [A]_{\delta-I}$ if and only if there exists a **net** $(\chi_d)_{d\in D}$ in A and $\chi_d \propto_{\delta-I} x$.

Proof :

→) Let $x \in [A]_{\delta-I}$, then $U \cap A \neq \emptyset$, for every **R-I-open** set U , $x \in U$. Notice that $(N_{R-I}(x), \subseteq)$ is a directed set , such that for all $U_1, U_2 \in N_{R-I}(x)$, $U_1 \geq U_2$ if and only if $U_1 \subseteq U_2$. Since for all $U \in N_{R-I}(x)$,$U \cap A \neq \emptyset$, then we can define a **net** $\chi : N_{R-I}(x) \to X$ as follows : $\chi(U) = \chi_U \in U \cap A$,$U \in N_{R-I}(x)$. To prove that $\chi_U \propto_{\delta-I} x$ .Let $B \in N_{R-I}(x)$, thus $B \cap U \in N_{\delta-I}(x)$. Since $B \cap U \subseteq U$, then $B \cap U \geq U$, $\chi(B \cap U) = \chi_{B\cap U} \in B \cap U \subseteq B$. Hence $\chi_U \propto_{\delta-I} x$.

(← Let $(\chi_d)_{d\in D}$ be a net in A , such that $\chi_d \propto_{\delta-I} x$, and let U be a **R-I-open** set ,$x \in U$. Since $\chi_d \propto_{\delta-I} x$, then $(\chi_d)_{d\in D}$ is frequently in U . Thus $U \cap A \neq \emptyset$, for all **R-I-open** set U, $x \in U$. Hence $x \in [A]_{\delta-I}$.

1-15 Remark : The space (X,T,I) is **δ-I-compact** if and only if the space $(X, T_{\delta-I}, I)$ is **compact** .

1-16 Remark : Every **compact** space is **δ-I-compact** space .

1-17 Theorem [10] :

(i) A **closed** subset of **compact** space is a **compact** .

(ii) In any space, the intersection of a **compact** set with a **closed** set is **compact** .

(iii) Every **compact** subset of **T_2**-space is a **closed** .

1-18 Proposition : Every **δ-I-closed** subset of **δ-I-compact** space is a **δ-I-compact**.

Proof : Let A be δ-I-closed in a δ-I-compact space (X,T,I) . By Remark (1-15) A is closed in a compact space $(X , T_{\delta\text{-}I} , I)$, by Theorem (1-17) A is a compact . hence A is δ-I-compact in (X,T,I) .

1-19 Proposition : An ideal topological space (X,T,I) is**δ-I-compact** if and only if every net in X has **δ-I-cluster** point in X.

Proof :

→) Let (X,T,I) be a **δ-I-compact** space and $(\chi_d)_{d\in D}$ be a **net** in X , then by Remark (1.15) , $(X , T_{\delta\text{-}I} , I)$ is a **compact** space . Then by Theorem (1.12) , the net $(\chi_d)_{d\in D}$ has **cluster** point x in $(X , T_{\delta\text{-}I} , I)$, then by Remark (1.11.iii) , x is a **δ-I-cluster** point of the net $(\chi_d)_{d\in D}$ (i.e. $\chi_d \propto_{\delta-I} x$).Hence every **net** in X has **δ-I-cluster** point in (X,T,I) .

←) Let every **net** in X has **δ-I-cluster** point in (X,T,I) , then by Remark (1.11.iii) , every **net** in X has **cluster** point in $(X , T_{\delta\text{-}I} , I)$ .Then by Theorem (1.11) , $(X , T_{\delta\text{-}I} , I)$ is a **compact** space , therefore by Remark (1.15) , (X,T,I) is a **δ-I-compact** space .

1-20 Definition : Let X be a space and $W \subseteq X$. We say that W is **compactly δ-I-closed** set if $W \cap K$ is **δ-I-compact** , for every **δ-I-compact** set K in X .

1.21 Proposition : In any space X :

(i) the intersection of a **δ-I-closed** set with a **δ-I-compact** set is a **δ-I-compact** set .

(ii) Every **δ-I-closed** subset of a space X is **compactly δ-I-closed** .

Proof :

(i) Let (X,T,*I*) be an ideal topological space and A,B are **δ-I-compact and δ-I-closed** , then A,B are **compact and closed** in (X , $T_{\delta\text{-}I}$, *I*). By Theorem (1.17.ii) A∩B is compact . Hence A∩B is **δ-I-compact** in (X,T,*I*).

(ii) Let A be a **δ-I-closed** subset of a space X and let K be a **δ-I-compact** set in X . Then by Proposition (1.21.i) , A∩K is a **δ-I-compact**.Thus A is **compactly δ-closed** set .

1-22 Remarks :

(i) Every **δ-I-closed** subset of a **compact** space is **δ-I-compact** .

(ii) Every **δ-I-compact** subset of a **T_2**-space is **δ-I-closed** .

Proof :

(i) By Remark (1-16) and Proposition(1-18).

(ii) By Remark (1-15) and Theorem (1-17-iii).

1-23 Theorem : Let X be a **T_2**-space . A subset A of X is a **compactly δ-I-closed** if and only if A is a **δ-I-closed** set .

Proof :

→) Let A be a **compactly δ-I-closed** set in X, since A⊆$[A]_{\delta\text{-}I}$ and let x ∈$[A]_{\delta\text{-}I}$. Then by Proposition (1.15) , there exists a **net** $(\chi_d)_{d\in D}$ in A , such that $\chi_d \xrightarrow{\delta-I} x$. Then F = {χ_d , x} is a **δ-I-compact** set . Since A is **compactly δ-I-closed** , then A∩F is a **δ-compact** set . But X is a **T_2**-space , then by Remark (1.22.ii) , $A\cap F$ is a **δ-I-closed** set . Since $\chi_d \xrightarrow{\delta-I} x$ and $\chi_d \in A\cap F$, then by Proposition (1.15) , $x \in A\cap F \Longrightarrow x \in A$. Hence $\mathbf{[A]}_{\delta\text{-}I} \subseteq A$, therefore A is a **δ-I-closed** set .

←) By Proposition (1.21.ii) .

<u>1-24 Proposition :</u>Let (X,T,*I*) be a **compact**, **T$_2$**-space and $A \subseteq X$. Then :

(i) A is **closed** if and only if A is **δ-I-closed** .

(ii) A is **compact** if and only if A is **δ-I-compact** .

<u>Proof :</u>

(i) →) Let A be a **closed** set in X . Since X is a **compact** space , by Theorem (1.17.i) , A is a **compact** set , so its **δ-I-compact** set . Now ,Since X is a **T$_2$**-space , then by Remark (1.22.ii) , A is a **δ-I-closed** set in X .

(← By Proposition (1.6.ii) .

(ii) →) Clear, since every **compact** set is **δ-I-compact** .

(← Let A is **δ-I-compact** set in X . Since X is **T$_2$**-space , then by Remark (1.22.ii) , A is a **δ-I-closed** set in X , and then it's a **closed** set . Since X is a **compact** space , then by Theorem (1.17.i) , A is a **compact** set in X .

<u>1-25 Definition :</u> A function f :X → Y is called a **δ-I-closed** function if the image of each **closed** subset of X is a **δ-I-closed** set in Y .

<u>1-26 Proposition :</u> Let X and Y be an ideal topological spaces , f : X → Y be a **δ-I-closed** function of X into Y . Then for each **clopen** subset T of Y , $f_T : f^{-1}(T) \rightarrow T$ is a **δ-I-closed** function .

<u>Proof :</u> Let F be a **closed** subset of $f^{-1}(T)$. Then there is a **closed** subset F_1 of X , such that $F = F_1 \cap f^{-1}(T)$. Since $f_T(F) = f(F_1)\cap T$, and $f(F_1)$ is a **δ-I-closed** in Y and T is **clopen** in Y then by proposition (1.6.i)and (1.6.iii), $f(F)\cap T$ is **δ-I-closed** in Y.By proposition (1.8) $f(F)\cap T$ is **δ-I-closed** in T . Thus f_T is a **δ-I-closed** function .

1-27 Proposition : Let X , Y and Z be an ideal topological spaces , f : X → Y be a **closed** function and g : Y → Z be a **δ-I-closed** function then go$f$: X → Z is a **δ-I-closed** function .

Proof : Let F be a **closed** subset of X , then f(F) is **closed** set in Y . But g is a **δ-closed** function , then g(f(F)) = (go$f$)(F) is a **δ-I-closed** set in Z . Then go$f$: X → Y is a **δ-I-closed** function

1-28 Definition : Let X and Y be an ideal topological spaces . We say that the function f : X → Y is a **δ-compact** function if the inverse image of each **δ-compact** set in Y, is a **compact** set in X.

1-29 Definition : Let X and Y be an ideal topological spaces and f : X → Y be a function , then f is called **δ-I-irresolute** function if f^{-1}(A) is a **δ-I-open** set in X for every **δ-I-open** set A in Y .

2- δ-I- Proper Function :

In this section, we introduce the definition of **δ-I-proper function** and study the relation between this type of maps and certain types of functions,such as(**δ-I-closed mapping** and **δ-I-compact mapping**).

Now we review some basic definitions , propositions and theorems about **proper function**.

2-1 Proposition [1] : Let X and Y be spaces , and f : X → Y be a function . Then f is called a **proper function** if :

(i) f is **continuous** function .

(ii) $f \times I_Z : X \times Z \to Y \times Z$ is **closed** , for every space Z .

Clearly , every **proper** function is **closed** function, and every **homeomorphism** is a **proper** function.

2-2 Proposition [1] : Let X and Y be spaces and f : X → Y be a **continuous** , **one to one**, function . Then the following statements are equivalent :

(i) f is **proper** .

(ii) f is **closed** .

(iii) f is a **homeomorphism** from X onto a **closed** subspace of Y.

2-3 Proposition [1] : Let X and Y be spaces , and f : X → Y be a **proper** function . If T is a subset of Y , then $f_T : f^{-1}(T) \rightarrow T$ is a **proper** function .

2-4 Proposition [1] : A space X is **compact** if and only if the function f : X → P = {w} is **proper** , where w any point which does not belong to X .

2-5 Theorem [1] : Let f :X → Y be a **continuous** function . Then the following statements are equivalent :

(i) f is a **proper** function .

(ii) f is a **closed** function and $f^{-1}(\{y\})$ is **compact** for each $y \in Y$.

(iii) If (χ_d) is a net in X and $y \in Y$ is a **cluster** point of the net $f(\chi_d)$,then there is a **cluster** point $x \in X$ of (χ_d),such that $f(x) = y$.

2-6 Definition : Let X and Y be an ideal topological spaces , and f : X → Y be a function . Then f is called a **δ-I-proper function** if :

(i) f is **continuous** .

(ii) $f \times I_Z : X \times Z \rightarrow Y \times Z$ is **δ-I-closed** , for every space Z .

2-7 Example :

2-8 Remarks :

(i) Every **δ-I-proper** function is **δ-I-closed** .

(ii)Every **δ-I-proper** function is **proper** .

2-9 Proposition : Let X and Y be an ideal topological spaces , and $f : X \to Y$ be a **δ-I-proper** function . If T is a **clopen** subset of Y , then $f_T : f^{-1}(T) \to T$ is a **δ-I-proper** function .

Proof : Since $f : X \to Y$ is a **continuous** function , then f_T is a **continuous** function . To prove that $f_T \times I_Z : f^{-1}(T) \times Z \to T \times Z$ is a **δ-I-closed** function , for every space Z . Notice that $f_T \times I_Z \equiv (f \times I_Z)_{T\times Z}$. Since T is a **clopen** subset of Y and Z is clopen set then, T×Z is a **clopen** subset of Y×Z , thus by Proposition (1.23) , $(f \times I_Z)_{T\times Z} \equiv (f_T \times I_Z)$ is a **δ-I-closed** function , hence $f_T : f^{-1}(T) \to T$ is a **δ-I-proper** function .

2-10 Theorem : Let X and Y be spaces , and $f : X \to Y$ be a **continuous** function . Then the following statements are equivalent :

(i) f is a **δ-I-proper** function .

(ii) f is a **δ-I-closed** function and $f^{-1}(\{y\})$is **compact** for each $y \in Y$.

(iii) If $(\chi_d)_{d\in D}$ is a net in X and $y \in Y$ is a **δ-I-cluster** point of $f(\chi_d)$, then there is a **cluster** point $x \in X$ of $(\chi_d)_{d\in D}$, such that $f(x) = y$.

Proof :

(i→ii).

Let $f : X \to Y$ be a **δ-I-proper** function , then $f \times I_Z : X \times Z \to Y \times Z$ is a **δ-I-closed** for every space Z . Let $Z = \{t\}$, then $X \times Z = X \times \{t\} \cong X$ and $Y \times Z = Y \times \{t\} \cong Y$, and we can replace $f \times I_Z$ by f , thus f is **δ-I-closed** . Now, let $y \in Y$. Since f is a **δ-I-proper** , then by Remark (2.8.ii) , f is **proper** function , so by Theorem (2.5) , $f^{-1}(\{y\})$ is **compact** for each $y \in Y$.

(ii → iii).

Let $(\chi_d)_{d\in D}$ be a net in X and $y \in Y$ be a **δ-I-cluster** point of a net $f(\chi_d)$ in Y .

Assume that $f^{-1}(y) \neq \emptyset$, if $f^{-1}(y) = \emptyset$, then $y \notin f(X) \to y \in (f(X))^c$, since X is a **closed** set in X and f is a **δ-I-closed** function , then $f(X)$ is a **δ-I-closed**

$\bigcup U_{x_2} \ldots \bigcup U_{x_n}$,then $f^{-1}(y) \bigcap [\bigcup_{i=1}^{n} U_{x_i}]^c = \emptyset \Longrightarrow f^{-1}(y) \bigcap [\bigcap_{i=1}^{n} U^c_{x_i}] = \emptyset$. But $(x_i)_{i\in\Lambda}$ is not frequently in U_{x_i} ,$\forall$ i = 1, … , n . Thus (χ_d) is not frequently in $\bigcup_{i=1}^{n} U_{x_i}$, but $\bigcup_{i=1}^{n} U_{x_i}$ is an **open** set in X , then $\bigcap_{i=1}^{n} U^c_{x_i}$ is a **closed** set in X . Thus $f(\bigcap_{i=1}^{n} U^c_{x_i})$ is a **δ-I-closed** set in Y .

Suppose $y \notin f(\bigcap_{i=1}^{n} U^c_{x_i})$, if $y \in f(\bigcap_{i=1}^{n} U^c_{x_i})$, then there exists $x \in \bigcap_{i=1}^{n} U^c_{x_i}$, such that $f(x) = y$, thus $x \notin \bigcup_{i=1}^{n} U_{x_i}$, but $x \in f^{-1}(y)$, therefore $f^{-1}(y)$ is not a subset of $\bigcup_{i=1}^{n} U_{x_i}$, and this is a contradiction . Hence there is an **R-I-open** set A in Y , such that $y \in A$ and $A \cap f(\bigcap_{i=1}^{n} U^c_{x_i}) = \emptyset \Longrightarrow f^{-1}(A) \cap f^{-1}(f(\bigcap_{i=1}^{n} U^c_{x_i})) = \emptyset \Longrightarrow f^{-1}(A) \cap [\bigcap_{i=1}^{n} U^c_{x_i}] = \emptyset \Longrightarrow f^{-1}(A) \subseteq \bigcup_{i=1}^{n} U_{x_i}$. But $(f(\chi_d))$is frequently in A , then $(\chi_d)$is frequently in $f^{-1}(A)$, and then (χ_d) is frequently in $\bigcup_{i=1}^{n} U_{x_i}$. This is contradiction , and this is complete the proof .

(iii → i) Let Z be any space . To prove that $f : X \to Y$ is a **δ-I-proper** function , i.e , to prove that $f \times I_Z : X \times Z \to Y \times Z$ is a **δ-I-closed** function . Let F be a **closed** set in X×Z . To prove that $(f \times I_Z)(F)$ is a **δ-I-closed** set in Y×Z .Let $(y,z) \in [(f \times I_z)(F)]_{\delta\text{-}I}$, then by Proposition (1.14) ,there exists a **net** $\{(y_d , z_d)\}_{d\in D}$ in $(f \times I_Z)(F)$ such that $(y_d , z_d) \propto_{\delta-I} (y,z)$, then $(y_d , z_d) = ((f \times I_Z)(x_d , y_d))$, where $\{(x_d , y_d)\}_{d\in D}$ is a **net** in F . Thus $(f(x_d) ,I_Z(z_d)) \propto_{\delta-I} (y,z)$, so $f(x_d) \propto_{\delta-I} y$ and $z_d \propto_{\delta-I} z$. Then by (iii),$\exists$ $x \in X$, such that $x_d \propto x$ and $f(x) = y$,Since $(x_d , z_d) \propto (x,z)$ and $\{(x_d , z_d)\}_{d\in D}$ is a **net** in F , thus $(x,y) \in Cl(F)$.

Since F=Cl(F),then $(x,y) \in F \Longrightarrow (y,z)=((f \times I_Z)(x,y)) \Longrightarrow (y,z) \in (f \times I_Z)(F)$, and then $[(f \times I_z)(F)]_{\delta\text{-}I} = (f \times I_Z)(F)$, thus $(f \times I_Z)(F)$ is a **δ-I-closed** set in Y×Z . Hence

$f \times I_Z : X \times Z \rightarrow Y \times Z$ is a **δ-I-closed** function , hence $f : X \rightarrow Y$ is a **δ-I-proper** function .

2-11 Theorem : Let $f : X \rightarrow P = \{w\}$ be a function on a space X .Then f is **δ-I-proper** if and only if X is a **compact** space, where w is any point which does not belong to X .

Proof :

$\rightarrow$) Let $f : X \rightarrow P = \{w\}$ be a **δ-I-proper** function on X , then by Remarks (2.8.ii), f is a **proper** function. By proposition (2.4) , X is a **compact** set .

($\leftarrow$ Let X be a **compact** space . Since P is a single point , then f is a **continuous** function. To prove that $f : X \rightarrow P = \{w\}$ is a **δ-proper** function:

(i) Since $f^{-1}(P) = X$, then $f^{-1}(P)$ is a **compact** set .

(ii) Let F is a **closed** subset of X, then either: $f(F) = \emptyset$ or $f(F) = \{w\}$. So $f(F)$ is **δ-I-closed** in P , then f is a **δ-I-closed** function . Thus by Theorem (2.10) , f is a **δ-I-proper** function.

2-12 Proposition : Let X and Y be spaces . If $f : X \rightarrow Y$ is a **δ-I-proper** function,then $f_{\{y\}} : f^{-1}(\{y\}) \rightarrow \{y\}$ is a **δ-I-proper** function,for all $y \in Y$.

Proof : Since $f : X \rightarrow Y$ is a **δ-I-proper** function , then $f^{-1}(\{y\})$ is **compact** for each $y \in Y$. Since $\{y\}$ is a single point , then by Theorem (2.10) ,$f_{\{y\}} : f^{-1}(\{y\}) \rightarrow \{y\}$ is a **δ-I-proper** function.

2-13 Proposition : Let X , Y and Z be spaces . If $f : X \rightarrow Y$ is **proper** and $g : Y \rightarrow Z$ is a **δ-I-proper** function, then $g o f : X \rightarrow Z$ is a **δ-I-proper** function .

Proof : To prove that $g o f : X \rightarrow Z$ is a **δ-I-proper** function:

(i) Since f :X → Y is a **proper** function, then f is **closed** . Similarly , since g : Y → Z is a **δ-proper** function , then g is **δ-I-closed** . Thus by Proposition (1.27) , go$f$: X → Z is a **δ-I-closed** function .

(ii) Let z ∈ Z , then $g^{-1}(\{z\})$ is a **compact** set in Y,and then $f^{-1}(g^{-1}(\{z\}) =(gof)^{-1}(\{z\})$ is a **compact** set in X . Therefore by (i) , (ii) and since go$f$ is **continuous** then by using Theorem (2.10) , go$f$ is a **δ-I-proper** function .

<u>2-14 Proposition :</u> Let X , Y and Z be spaces , and f : X → Y and g : Y → Z be **continuous** functions, such that go$f$: X → Z is a **δ-I-proper** function . If f is **onto** , then g is a **δ-I-proper** function.

<u>Proof :</u>

(i) Let F be a **closed** subset of Y,since f is **continuous**,then $f^{-1}(F)$ is **closed** in X . Since go$f$ is a **δ-proper** function , then $gof(f^{-1}(F))$ is **δ-I-closed** in Z . But f is **onto** , then $gof(f^{-1}(F)) = g(F)$.Hence g(F) is a **δ-I-closed** set in Z . Thus g is a **δ-I-closed** function .

(ii) Let z ∈ Z , since go$f$ is a **δ-I-proper** function, then by Theorem (2.10) , the set$(gof)^{-1}(\{z\}) = f^{-1}(g^{-1}(\{z\}))$ is **compact** . Now , since f is **continuous** , then $f(f^{-1}(g^{-1}(\{z\})))$ is **compact** set , but f is **onto** , then $f(f^{-1}(g^{-1}(\{z\}))) = g^{-1}(\{z\})$ is **compact** for every z ∈ Z . So by Theorem (2.10) , the function g is **δ-I-proper** .

<u>2-15 Proposition :</u> Let X , Y and Z be spaces , and f : X → Y , g : Y → Z be **continuous** functions,such that go$f$: X → Z is a **δ-I-proper** function . If g is **one to one** ,**δ-I-irresolute** function then f is a **δ-I-proper** function.

<u>Proof :</u>

(i) Let F be a **closed** subset of X . Then $(gof)(F)$ is an **δ-I-closed** set in Z . Since g : Y → Z is **one to one** , **δ-I-irresolute** function, then $g^{-1}(g(f(F))) = f(F)$ is **δ-I-closed** in Y . Hence the function f : X → Y is **δ-I-closed** .

(ii) Let $y \in Y$, then $g(y) \in Z$.Since $g\circ f : X \to Z$ is **δ-I-proper** and g is **one to one**,then the set $(g\circ f)^{-1}(g(\{y\})=f^{-1}(g^{-1}(g(\{y\}))) = f^{-1}(\{y\})$ is **compact**,for every $y \in Y$. Therefore by Theorem (2.10),the function $f : X \to Y$ is **δ-I-proper**.

2-16 Proposition : Let X , Y and Z be spaces , $f : X \to Y$ be a **continuous** function and $g : Y \to Z$ be a **δ-I-irresolute** function,such that $g\circ f : X \to Y$ is a **δ-I-proper** function.If Y is a **T_2**-space,then f is **δ-I-proper** .

Proof : Consider the commutative diagram :

$$\begin{tikzcd}
X \arrow[r, "\psi"] \arrow[d, "f"'] & X\times Y \arrow[d, "(g\circ f)\times I_Y"] \\
Y \arrow[r, "K"'] & Z\times Y
\end{tikzcd}$$

$\psi(x) = (x,f(x))$ and $K(y) = (g(y), y)$. Since Y is **T_2**-space , then the graph of ψ is **closed** in $X\times Y$ [1, Proposition .5.P.99] , and since ψ is **one to one** , then by Proposition (2.2),ψ is a **proper** function . We have $(g\circ f)\times I_Z$ is **δ-I-proper**, then by Proposition (2.13),$((g\circ f)\times I_Z)\circ\psi$ is **δ-I-proper** . But $((g\circ f)\times I_Z)\circ\psi = K\circ f$, so that $K\circ f$ is **δ-I-proper** . Since g is a **δ-I-irresolute** function , then K is **δ-I-irresolute** . Therefore by Proposition (2.15) , f is a **δ-I-proper** function .

2-17 Corollary : Every **continuous** function of a compact space X into a **T_2**-space Y is **δ-I-proper** .

Proof : Let $f : X \to Y$ be a **continuous** function .To prove that f is **δ-I-proper** . Let $g : Y \to P$ be a function (where P is a singleton set) , since X is a **compact** space , then by Theorem (2.11) , $g\circ f : X \to P$ is **δ-I-proper** . Since Y is a **T_2-space** , then by Proposition (2.16) , f is **δ-I-proper** function .

2-18 Proposition : Let $f : X \to Y$ be a **δ-I-proper** function , then $f\times I_Z : X\times Z \to Y\times Z$ is a **δ-proper** function , for every space Z .

Proof : Since f is **δ-I-proper** function , then $f\times I_W$ is a **δ-I-closed** function , for every space W . Notice that $f\times I_Z\times I_W = f\times I_{Z\times W}$, but $f\times I_{Z\times W}$ is a **δ-I-closed** function , then $f\times I_Z\times I_W$ is **δ-I-closed** , for every space W . Hence $f\times I_Z$ is **δ-I-proper** function .

2-19 Proposition : Let X be a **compact** space and Y be any topological space , then the **projection** function $Pr_2 : X\times Y \to Y$ is **δ-I-proper** .

Proof : Consider the commutative diagram :

$$\begin{tikzcd}
X\times Y \arrow[rr, "f\times I_Y"] \arrow[dr, "Pr_2"'] & & \{p\}\times Y \arrow[dl, "h(\cong)"] \\
& Y &
\end{tikzcd}$$

Where $h : \{p\}\times Y \to Y$ is the **homeomorphism** of $\{p\}\times Y$ **onto** Y , such that $p \notin X$ and $Pr_2 : X\times Y \to Y$ is the projection of $X\times Y$ into Y . Since X is a **compact** space , then by Theorem (2.11) , $f : X \to \{p\}$ is **δ-I-proper** and $I_Y : Y \to Y$ is a **proper** function , then by Proposition (2.18) , $f\times I_Y$ is a **δ-I-proper** function . Hence $ho(f\times I_Y)$ is a **δ-I-proper** function , but $Pr_2 = ho(f\times I_Y)$, then Pr_2 is a **δ-I-proper** function .

2-20 Proposition : Let X and Y be spaces . If $f : X \to Y$ is a **δ-I-proper** function , then f is a **δ-I-compact** function .

Proof : Let A be a **δ-I-compact** subset of Y . To prove that $f^{-1}(A)$ is a **compact** set in X , let $(\chi_d)_{d\in D}$ be a **net** in $f^{-1}(A)$, then $f(\chi_d)$ is a **net** in A . Since A is a **δ-I-compact** set in Y , then by Proposition (1.14) , there $y \in A$, such that y is a **δ-I-cluster** point of $f(\chi_d)$. Since f is **δ-I-proper** , then by

Theorem (2.10) , there exists $x \in X$, such that x is a **cluster** point of$(\chi_d)$and $f(x) = y$ .Then $x \in f^{-1}(A)$. Thus every **net** in $f^{-1}(A)$ has **cluster** point in itself , then by Theorem (1.12) , $f^{-1}(A)$ is a **compact** set in X . Therefore $f : X \rightarrow Y$ is a **δ-I-compact** function .

2-21 Theorem : Let X and Y be spaces , such that Y is a **T_2**-space . If $f : X \rightarrow Y$ is a **continuous** function , then f is a **δ-I-proper** function if and only if f is a **δ-I-compact** function .

Proof :

→) By Proposition (2.20) .

←) To prove that f is a **δ-I-proper** function :

(i) Let F be a **closed** subset of X . To prove that $f(F)$ is a **δ-I-closed** set in Y, let K be a **δ-I-compact** set in Y , then $f^{-1}(K)$ is a **compact** set in X , then by Theorem (1.17.ii) , $F \cap f^{-1}(K)$ is **compact** in X . Since f is **continuous** , then $f(F \cap f^{-1}(K))$ is **compact** set in Y , and then its **δ-I-compact** . But $f(F \cap f^{-1}(K)) = f(F) \cap K$, then $f(F) \cap K$ is **δ-I-compact** , thus $f(F)$ is **compactly δ-I-closed** set in Y . Since Y is a **T_2**-space , then by Theorem (1.23) , $f(F)$ is a **δ-I-closed** set in Y. Hence f is a **δ-I-closed** function .

(ii) Let $y \in Y$, then $\{y\}$ is **δ-I-compact** in Y . Since f is a **δ-I-compact** function , then $f^{-1}(\{y\})$ is **compact** in X , therefore by Theorem (2.10),f is a **δ-I-proper** function .

2-22 Theorem : Let $f : X \rightarrow P = \{w\}$ be a function on a space X , where w is any point which does not belong to X , then the following statements are equivalent :

(i) f is a **δ-compact** function .

(ii) f is a **δ-proper** function .

(iii) f is a **proper** function .

(iv) X is a **compact** space .

Proof :

(i → ii).

By Theorem (2.21) .

(ii → iii).

By Remark (2.8.ii) .

(iii → iv).

By Proposition (2.4) .

(iv → i).

Since $f^{-1}(P) = X$ and X is a **compact** space , then f is a **δ-I-compact** function .

2-23 Theorem : Let X and Y be spaces , such that Y is a **compact** , **T_2**-space and $f : X \to Y$ be a **continuous** mapping , then the following statements are equivalent :

(i) f is a **proper** function .

(ii) f is a **compact** function .

(iii) f is a **δ-I-compact** function .

(iv) f is a **δ-I-proper** function .

Proof :

(i → ii).

Let K be a **compact** set in Y . To prove that $f^{-1}(K)$ is **compact** set in X . Let $f_K : f^{-1}(K) \to K$ and $g : K \to P$. Since f is **proper** , then by Proposition (2.3) , f_K is **proper** and by Proposition (2.4) , g is a **proper** function , then by

[8. Proposition 5 .P.99] , $go f_K : f^{-1}(K) \to P$ is a **proper** mapping , so by Proposition (2.4) , $f^{-1}(K)$ is a **compact** set . Hence f is a **compact** function .

(ii → iii).

Let H be a **δ-I-compact** set in Y . To prove that $f^{-1}(H)$ is **compact** in X . Since Y is a **compact** , **T_2**-space , then by Proposition (1.24.ii) , H is a **compact** set in Y , then by (ii) , $f^{-1}(H)$ is a **compact** set in X. Hence f is an **δ-compact** function .

(iii → iv) By Theorem (2.21) .

(iv → i) By Remark (2.8.ii) .

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