



Superconvergence of nonconforming finite element approximation for the second order elliptic problem

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ABSTRACT

In this paper a general superconvergence of nonconforming finite element method for the second order elliptic problem is derived. In order to verify and support the theoretical results numerical examples are given.

Keywords: nonconforming, finite element, superconvergence, elliptic problem.

1. Introduction

“The superconvergence of finite element (FE) solutions is an interesting and useful phenomenon in the scientific computing of real world problems and has become an area of active research in recent years” [4]. Wang [9] proposed and analyzed the L^2 -projection method for the least-squares conforming finite element method on the second order elliptic problem. The goal of this article is to derive a general superconvergence of nonconforming FE with its application for elliptic problem by applying certain postprocessing.

The paper is organized as follows. In section two, we give a preliminaries for the nonconforming finite element. In section three, we derived some results to improve the existing accuracy with framework for the procedure that we given in section two. Finally, in section four, several numerical examples are tested to support and confirm the theoretical results derived in section three.

2. Preliminaries

We shall consider the following elliptic problem with Dirichlet boundary condition. This model problem seeks an unknown function u such that

$$-\Delta u + u = f \quad \text{in } \Omega \quad 2.1$$

$$u = 0 \quad \text{on } \partial\Omega, \quad 2.2$$

where Δ is the Laplacian operator and f is a given function. Let Ω be an open domain in \mathbb{R}^2 with a Lipschitz boundary $\partial\Omega$. Let $H^s(\Omega)$ be the Sobolev spaces and their associated inner products $(\cdot, \cdot)_{s,\Omega}$ and norms $\|\cdot\|_{s,\Omega}$, $s \geq 0$. Let $L^2(\Omega)$ be a coincides space of square integrable functions, [4].

A variational formulation of (2.1)–(2.2) is to find $u \in H_0^1(\Omega)$ such that

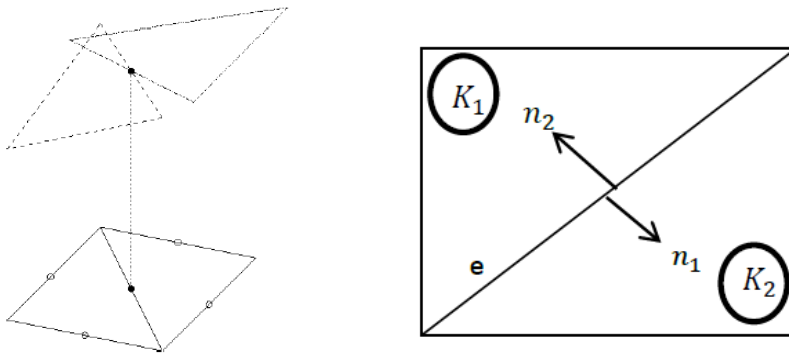
$$a(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega),$$

where

$$a(u, v) = (\nabla u, \nabla v) + (u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Omega} u \cdot v \, dx$$

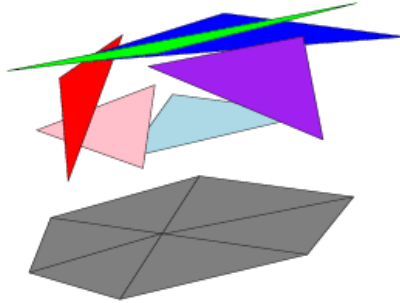
Let \mathcal{T}_h be a partition of the domain Ω into $\Omega = \cup_{K \in \mathcal{T}_h} K$. Suppose that \mathcal{T}_h is quasiuniform, that is it the inverse assumption is satisfy and regular [8],

Let \mathcal{E}_h denote the union of the all boundaries of all elements K of \mathcal{T}_h and $\mathcal{E}_h^0 = \mathcal{E}_h \cap \partial\Omega$, the collection of all interior edges. Let e be an interior edge shared by two elements K_1 and K_2 in \mathcal{T}_h , let n_1 be unit normal vectors on e pointing exterior to K_2 and n_2 be unit normal vector on e pointing exterior to K_1 , see Figure 2.1.



Define the jump of a function v across an edge e as $[v] = v|_{\partial K_2} - v|_{\partial K_1}$. The nonconforming FE space associated with the mesh \mathcal{T}_h is defined as

$$V_h = \{v \in L^2(\Omega) : v|_K \in P_k(K), \forall K \in \mathcal{T}_h, v \text{ is continuous at middle point } e \in \mathcal{E}_h^0, v(m) = 0 \text{ such that } m \in \partial\Omega\}, \text{ see Figure 2.2.}$$



The nonconforming FE of (2.1) –(2.2) is to find $u_h \in V_h$

such that

$$a_h(u_h, v) = (f, v) \quad \forall v \in V_h \quad 2.3$$

where

$$a_h(u, v) = \sum_{K \in \mathcal{T}_h} \int_K \nabla u \cdot \nabla v \, dx + \sum_{K \in \mathcal{T}_h} \int_K u \cdot v \, dx.$$

Define the norm

$$\|v\|_{1,h} = \left(\sum_{K \in \mathcal{T}_h} \|v\|_{1,K}^2 \right)^{\frac{1}{2}},$$

where

$$\|v\|_{1,K}^2 = \int_K \nabla v \cdot \nabla v \, dx + \int_K v \cdot v \, dx.$$

3. Superconvergence by L^2 -projection

The L^2 -Projection is a postprocessing technique defined by Wang [9] for Galerkin methods. The idea is to project the FE solution to another finite element space with a different mesh size. Let \mathcal{T}_τ be a coarse mesh where $\tau \gg h$. Suppose that τ and h satisfying:

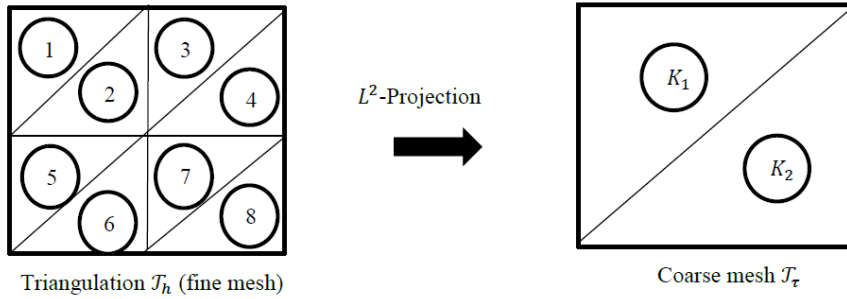
$$\tau = h^\alpha \quad 3.1$$

with $\alpha \in (0,1)$.

We will be seen that α plays an important role in the postprocessing. For now, let $V_\tau \subset H^{s-2}(\Omega)$, for the exact solution u . Define Q_τ to be the L^2 -projectors from $L^2(\Omega)$ onto the FE space V_τ .

We shall give the structure of the coarse mesh by L^2 -projection method.

Figure 3.1, shows a FE discretization of Ω .



$$C2F_{\text{mesh}} = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 4 & 6 & 7 & 8 \end{pmatrix} \begin{matrix} K_1 \\ K_2 \end{matrix}$$

Matrix coarse to fine mesh

Figure 3.1: Typical two dimensional discretization structure coarse mesh.

$$u_h \approx Q_\tau u_h .$$

The L^2 -projection $Q_\tau u_h$ satisfying

$$(Q_\tau u_h, v) = (u_h, v) \quad \forall v \in V_\tau,$$

then we have

$$(u_h - Q_\tau u_h, v) = 0 \quad \forall v \in V_\tau.$$

Since V_τ defined as follows:

$$V_\tau = \{v \in L^2(\Omega): v|K \in P_2(K), \forall K \in \mathcal{T}_\tau\}.$$

$$\text{So, } v(x) = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 \quad a_i \in \mathbb{R}, \quad i = 1, \dots, 6.$$

In order to compute the $Q_\tau u_h$, we first note that

$$(Q_\tau u_h, \varphi_i) = (u_h, \varphi_i), \quad \varphi_i \in P_2(K) \quad 3.2$$

where φ_i is the hat basis function spanning V_τ . Further, expanding $Q_\tau u_h$, it can be written as the linear combination

$$Q_\tau u_h = \sum_{j=1}^N c_j \varphi_j, \quad 3.3$$

where c_j , is N unknown coefficients, and substituting 3.3 into 3.2, we finally get the following system which given by:

$$\sum_{j=1}^N c_j (\varphi_j, \varphi_i) = (u_h, \varphi_i) \quad i = 1, \dots, N, \quad 3.4$$

the linear system of 3.4 can be written as

$$MX = D,$$

where $M = (m_{ij})$ is the $N \times N$ matrix with elements

$$\begin{aligned} m_{ij} &= (\varphi_j, \varphi_i), \\ m_{ij} &= \sum_{K \in \mathcal{T}_h} m_{ij}^K, \end{aligned}$$

where

$$m_{ij}^K = \int_K \varphi_j \varphi_i \, dx,$$

note that the subscript in m_{ij}^K is the local index, while in m_{ij} it is global index,

where $X = c_j = c_1, \dots, c_N$ is the solution vector with $c_j = Q_\tau u_h(N_j)$, $j = 1, \dots, N$ and $D = d_i = d_1, \dots, d_N$ is the force vector with

$$d_i^K = (u_h, \varphi_i) = (\sum_{i=1}^3 c_i \psi_i, \varphi_i),$$

where ψ_i is the local basis function of element K

$$d_i = \sum_{K \in \mathcal{T}_h} d_i^K,$$

where

$$d_i^K = \int_K u_h \varphi_i dK = (u_h, \varphi_i) = (\sum_{i=1}^3 c_i \psi_i, \varphi_i),$$

The element matrix of force vector (the entries of the 6×1) is given by:

$$d_i^K = \int_K u_h \varphi_i dK = (u_h, \varphi_i) = (c_1 \psi_1 + c_2 \psi_2 + c_3 \psi_3, \varphi_i),$$

then

$$d_i^K = \begin{bmatrix} (c_1 \psi_1 + c_2 \psi_2 + c_3 \psi_3, 1) \\ (c_1 \psi_1 + c_2 \psi_2 + c_3 \psi_3, x) \\ (c_1 \psi_1 + c_2 \psi_2 + c_3 \psi_3, y) \\ (c_1 \psi_1 + c_2 \psi_2 + c_3 \psi_3, x^2) \\ (c_1 \psi_1 + c_2 \psi_2 + c_3 \psi_3, xy) \\ (c_1 \psi_1 + c_2 \psi_2 + c_3 \psi_3, y^2) \end{bmatrix}$$

Also, the matrix of element K (the entries of the 6×6) given by:

$$m_{ij}^K = \begin{bmatrix} a_{11} & \cdots & a_{61} \\ \vdots & \ddots & \vdots \\ a_{16} & \cdots & a_{66} \end{bmatrix}$$

$$= \begin{bmatrix} (1,1) & (x, 1) & (y, 1) & (x^2, 1) & (xy, 1) & (y^2, 1) \\ (1,x) & (x, x) & (y, x) & (x^2, x) & (xy, x) & (y^2, x) \\ (1,y) & (x, y) & (y, y) & (x^2, y) & (xy, y) & (y^2, y) \\ (1,x^2) & (x, x^2) & (y, x^2) & (x^2, x^2) & (xy, x^2) & (y^2, x^2) \\ (1,xy) & (x, xy) & (y, xy) & (x^2, xy) & (xy, xy) & (y^2, xy) \\ (1,y^2) & (x, y^2) & (y, y^2) & (x^2, y^2) & (xy, y^2) & (y^2, y^2) \end{bmatrix}$$

Since

$$M = m_{ij} = \sum_{K \in \mathcal{T}_h} m_{ij}^K,$$

and the force vector

$$D = d_i = \sum_{K \in \mathcal{T}_h} d_i^K,$$

then $Q_\tau u_h$ is given by:

$$Q_\tau u_h = M^{-1}D,$$

where M^{-1} is the inverse of M .

We will prove the following lemma that we need it later.

Lemma 3.1: There is a constant, C , such that

$$|a(w, v)| \leq C \|w\|_{1,h} \|v\|_{1,h} \quad \forall w, v \in H_0^1(\Omega) \quad 3.5$$

Where C is independent of h .

Proof: Since the problem (2.1)–(2.2) satisfy the weak form

$$a(w, v) = (f, v) \quad \forall v \in H_0^1(\Omega),$$

where

$$a(w, v) = (\nabla w, \nabla v) + (w, v) = \int_{\Omega} \nabla w \cdot \nabla v \, dx + \int_{\Omega} wv \, dx.$$

An application of Cauchy–Schwarz inequality reveals

$$\begin{aligned} |a(w, v)| &= \left| \int_{\Omega} \nabla w \cdot \nabla v \, dx \right| + \left| \int_{\Omega} wv \, dx \right| \\ &\leq \|\nabla w\| \|\nabla v\| + \|w\| \|v\|. \end{aligned}$$

By the Poincare inequality we have

$$\begin{aligned} |a(w, v)| &\leq \|\nabla w\| \|\nabla v\| + \hat{C} \|\nabla w\| \hat{C} \|\nabla v\| \\ &\leq \|\nabla w\| \|\nabla v\| + \hat{C}^2 \|\nabla w\| \|\nabla v\| \\ &\leq (1 + \hat{C}^2) \|\nabla w\| \|\nabla v\| \\ &\leq C \|\nabla w\| \|\nabla v\|, \end{aligned}$$

where $C = 1 + \hat{C}^2$.

Since $\|\nabla w\| \leq \|w\|_{1,h}$

where $\|\nabla w\| = \left(\int_{\Omega} |\nabla w|^2 dx\right)^{\frac{1}{2}}$.

We have $|a(w, v)| \leq C\|w\|_{1,h}\|v\|_{1,h}$.

The following theorem can be found in [11].

Theorem 3.2: : Let u and u_h be the solutions of (2.1)–(2.2) and (2.3), respectively. Then, there is a constant C , such that

$$\|u - u_h\|_{1,h} \leq Ch^k \|u\|_{k+1}.$$

where C is an independent of h .

The superconvergence analysis requires certain regularity for the second order elliptic problem. To this end, suppose that the domain is so regular, that is $H^s, s \geq 1$ regularity for solution u , that is $\|u\|_s \leq \|f\|_{s-2} \quad \forall f \in H^{s-2}(\Omega), s \geq 1.$ 3.6

Define the finite element space $V_{\tau} \subset H^{s-2}(\Omega)$, for u .

Next, we will prove the following lemma.

Lemma 3.3: Suppose that (3.6) hold with $1 \leq s \leq k + 1$ and $V_{\tau} \subset H^{s-2}(\Omega)$. Then, there is a constant, C such that

$$\|Q_{\tau}u - Q_{\tau}u_h\| \leq Ch^{s-1+\alpha \min(0,2-s)} \|u - u_h\|_{1,h}. \quad 3.7$$

Where C independent of h and τ .

Proof: From the definition of $\|\cdot\|$ and Q_τ , we have

$$\|Q_\tau u - Q_\tau u_h\| = \sup_{\emptyset \in L^2(\Omega), \|\emptyset\|_0=1} |(Q_\tau u - Q_\tau u_h, \emptyset)| \quad 3.8$$

Using the definition of the L^2 -projection for Q_τ , we obtain

$(Q_\tau u - Q_\tau u_h, \emptyset) = (u - u_h, Q_\tau \emptyset)$, It follows,

$$\|Q_\tau u - Q_\tau u_h\| = \sup_{\emptyset \in L^2(\Omega), \|\emptyset\|_0=1} |(u - u_h, Q_\tau \emptyset)| \quad 3.9$$

Consider the following problem

$$-\Delta w + w = Q_\tau \emptyset \quad \text{in } \Omega \quad 3.10$$

$$w = 0 \quad \text{on } \partial\Omega, \quad 3.11$$

Multiplying (1.1) by v and integrating it over Ω , we have

$$a_h(u, v) = (f, v) + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial u}{\partial n} v ds, \quad 3.12$$

The difference of (2.3) and (3.12), then

$$a_h(u - u_h, v) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial u}{\partial n} v ds, \quad \forall v \in V_h \quad 3.13$$

Multiplying (3.10) by $u - u_h$, integrating it over Ω and by using (3.12), we have

$$(u - u_h, Q_\tau \emptyset) = (-\Delta w + w, u - u_h) = a_h(u - u_h, w) - \sum_{K \in \mathcal{T}_h} \int_{\partial K} (u - u_h) \frac{\partial w}{\partial n} ds$$

$$\begin{aligned}
 &= a_h (u - u_h, w - w_I) + a_h (u - u_h, w_I) - \sum_{K \in \mathcal{T}_h} \int_{\partial K} (u - u_h) \frac{\partial w}{\partial n} ds \\
 &= a_h (u - u_h, w - w_I) - \sum_{K \in \mathcal{T}_h} \int_{\partial K} (u - u_h) \frac{\partial w}{\partial n} ds \\
 &+ \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial u}{\partial n} w_I ds
 \end{aligned}$$

From [10], we have

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial u}{\partial n} w_I ds \leq Ch^{k+s-1} \|u\|_{k+1} \|w\|_s, \tag{3.14}$$

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} (u - u_h) \frac{\partial w}{\partial n} ds \leq Ch^{k+s-1} \|u\|_{k+1} \|w\|_s, \tag{3.15}$$

By using Cauchy Schwarz , (3.14) , (3.15) and (3.6) we get

$$\begin{aligned}
 |(u - u_h, Q_\tau \phi)| &= a_h (u - u_h, w - w_I) - \sum_{K \in \mathcal{T}_h} \int_{\partial K} (u - u_h) \frac{\partial w}{\partial n} ds \\
 &+ \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial u}{\partial n} w_I ds \\
 &\leq \|w - w_I\|_{1,h} \|u - u_h\|_{1,h} + Ch^{k+s-1} \|u\|_{k+1} \|w\|_s \\
 &\leq Ch^{k+s-1} \|u\|_{k+1} \|w\|_s \\
 &\leq Ch^{k+s-1} \|u\|_{k+1} \|Q_\tau \phi\|_{s-2} \|u\|_{k+1} \\
 &\leq Ch^{k+s-1} \tau^{\min(0,2-s)} \|Q_\tau \phi\| \|u\|_{k+1} \\
 &\leq Ch^{k+s-1+\alpha \min(0,2-s)} \|u\|_{k+1}
 \end{aligned} \tag{3.16}$$

From (3.8) and (3.16) it follows

$$\|Q_\tau u - Q_\tau u_h\| \leq Ch^{k+s-1+\alpha \min(0,2-s)} \|u\|_{k+1} \quad 3.17$$

Now, we can to estimate $u - Q_\tau u_h$.

Theorem 3.4: Suppose that (3.6) hold with $1 \leq s \leq k + 1$ and $V_\tau \subset H^{s-2}(\Omega)$. If the exact $\in H^{k+1}(\Omega) \cap H^{r+1}(\Omega) \cap H_0^1(\Omega)$, then there is a constant, C, such that

$$\begin{aligned} & \|u - Q_\tau u_h\| + h^\alpha \|\nabla_\tau(u - Q_\tau u_h)\| \\ & \leq Ch^{\alpha(r+1)} \|u\|_{r+1} + Ch^\sigma \|u - u_h\|_{1,h}, \end{aligned} \quad 3.18$$

where $\sigma = s - 1 + \alpha \min(0,2 - s)$.

Proof: By the definition Q_τ and (2.11), we have

$$\|u - Q_\tau u\| \leq C\tau^{r+1} \|u\|_{r+1} \leq Ch^{\alpha(r+1)} \|u\|_{r+1} \quad 3.19$$

Triangle inequality and combining (3.18) with (3.17) gives

$$\begin{aligned} & \|u - Q_\tau u_h\| \leq \|u - Q_\tau u\| + \|Q_\tau u - Q_\tau u_h\| \\ & \leq Ch^{\alpha(r+1)} \|u\|_{r+1} + Ch^{s-1+\alpha \min(0,2-s)} \|u - u_h\|_{1,h}, \end{aligned}$$

which completes the estimate for $\|u - Q_\tau u_h\|$.

Next, we estimate $\|\nabla_\tau(u - Q_\tau u_h)\|$ by following the similar idea in [4], then

$$h^\alpha \|\nabla_\tau(u - Q_\tau u_h)\| \leq Ch^{\alpha(r+1)} \|u\|_{r+1} + Ch^{k+s-1+\alpha \min(0,2-s)} \|u - u_h\|_{1,h}.$$

This completes the proof of the theorem

The above error estimate is optimized if α is selected as

$$\alpha(r + 1) = k + s - 1 + \alpha \min(0, 2 - s) \quad 3.20$$

Solving α from above yields

$$\alpha = \frac{k+s-1}{r+1-\min(0,2-s)}. \quad 3.21$$

4. Numerical Examples

We present several numerical experiments to conform Theorem 3.4. The triangulation of \mathcal{T}_h is describe by: (1) the domain is divided into an $n^3 \times n^3$ rectangular partition and (2) the diagonal line is connecting with the positive slope. Let $h = \frac{1}{n^3}$ as the mesh size. In the implementation, the quantity $Q_\tau u_h$ is L^2 -projection of u_h to V_τ associate with \mathcal{T}_τ . We define V_τ as follows:

$$V_\tau = \{v \in L^2(\Omega): v|_K \in P_2(K), \forall K \in \mathcal{T}_\tau\}.$$

By using Theorem 3.4, we get the order $O\left(h^{\frac{4}{3}}\right)$ for $\|\nabla_\tau(u - Q_\tau u_h)\|$. The following experiments results meet with proved theory.

Example 4.1: Let the domain $\Omega = [0,1] \times [0,1]$. Also, the real solution is assumed to be

$$u = x^5 \sin y$$

Table 4.1 shows that after the use of the postprocessing, the errors are reduced. The error in the H_1 -norm has a higher order, which is shown as $O(h^{1.3})$ for $\|\nabla_\tau(u - Q_\tau u_h)\|$, (see Figure 4.1). Figure 4.2 (a) and (b) give results for the FE approximation of the problem given in (1.1)–(1.2), both before and after post-processing. These results confirm with the theoretical result.

h	$\ u - u_h\ _1$	$\ u - Q_\tau u_h\ _1$
2^{-3}	0.2779e-1	0.9578e-2
3^{-3}	0.1182e-1	0.2924e-2
4^{-3}	0.6064e-2	0.1178e-2
5^{-3}	0.3511e-2	0.5636e-3
$O(h^k), k =$	0.9950	1.3624

Table 4.1: Errors on meshes \mathcal{T}_h and \mathcal{T}_τ

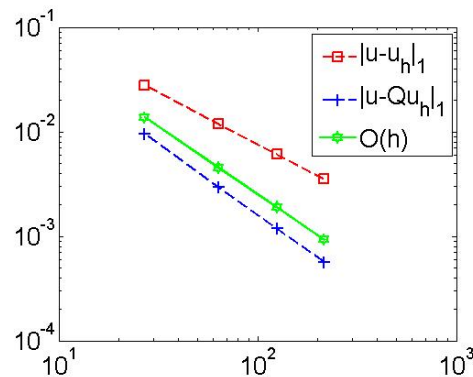


Figure 4.1: Convergence rate of error in Example 4.1. H_1 -norm error.

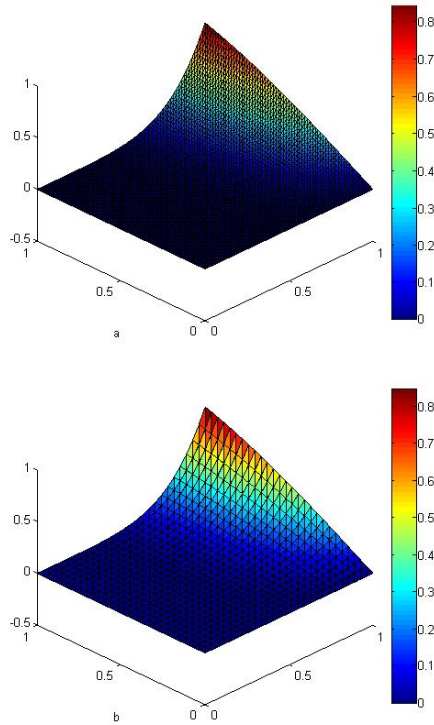


Figure 4.2: Results for $u = x^5 \sin y$ in Example 4.1. (a) Surface of u_h . (b) Surface of $Q_\tau u_h$.

Example 4.2: Let the domain $\Omega = [0,1] \times [0,1]$. Also, the real solution is assumed to be

$$u = (x - x^2)(y - y^2)$$

h	$\ u - u_h\ _1$	$\ u - Q_\tau u_h\ _1$
2^{-3}	0.2056e-2	0.1371e-2
3^{-3}	0.8679e-3	0.4336e-3
4^{-3}	0.4444e-3	0.1778e-3
5^{-3}	0.2571e-3	0.8580e-4
$O(h^k), k =$	0.9997	1.3329

Table 4.2: Errors on meshes \mathcal{T}_h and \mathcal{T}_τ .

From the results shown in Table 4.2, it is easy to note that the solution u in the H_1 -norm has the faster convergence, (see Figure 4.3). Figure 4.4 (a) and (b), shows that the approximate solutions u_h and $Q_\tau u_h$. This is in agreement with the previously stated theory.

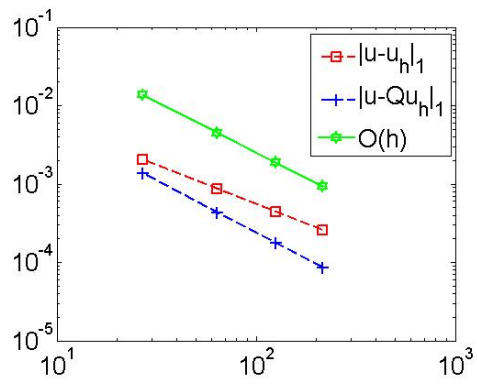


Figure 4.3: Convergence rate of error in Example 4.2. H_1 -norm error.

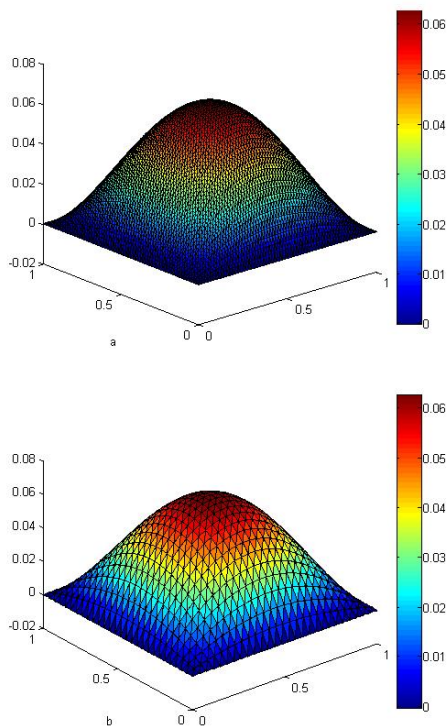


Figure 4.4: Results for $u = (x - x^2)(y - y^2)$ in Example 4.2. (a) Surface of u_h . (b) Surface of $Q_\tau u_h$.

Example 4.3: Let the domain $\Omega = [0,1] \times [0,1]$. Also, the real solution is assumed to be

$$u = (e^x - x^5) \sin(y)$$

h	$\ u - u_h\ _1$	$\ u - Q_\tau u_h\ _1$
2^{-3}	0.1960e-1	0.8396e-2
3^{-3}	0.8345e-2	0.2620e-2
4^{-3}	0.4281e-2	0.1065e-2
5^{-3}	0.2478e-2	0.5111e-3
$O(h^k), k =$	0.9944	1.3460

Table 4.3: Errors meshes \mathcal{T}_h and \mathcal{T}_τ

Table 4.3 gives the error profile for Example 4.3. Note that the gradient estimate is of order $O(h^{1.3})$ has the superconvergence, (see Figure 4.5). Figure 4.6 (a) and (b), shows that the approximate solutions u_h and $Q_\tau u_h$. Also, the approximate results are highly consistent with the theoretical L^2 -projection.

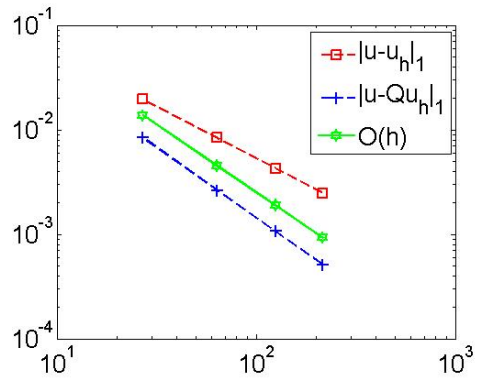


Figure 4.5: Convergence rate of error in Example 4.3. H_1 -norm error.

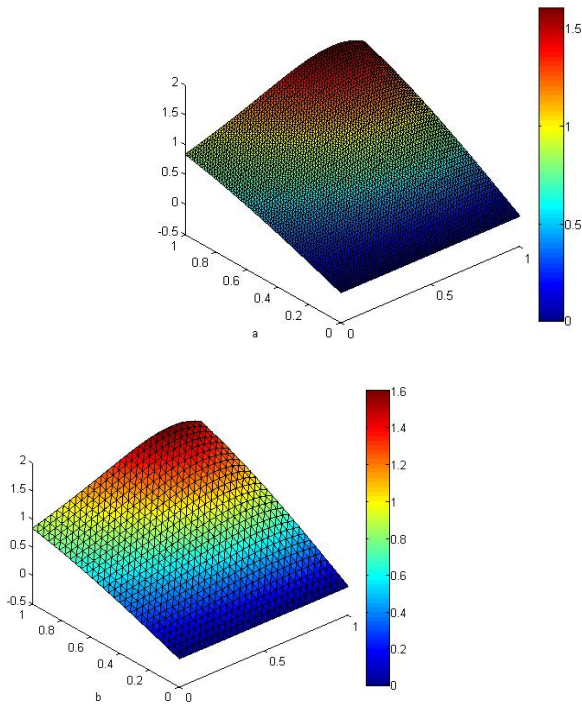


Figure 4.6: Results for $u = (e^x - x^5) \sin(y)$ in Example 4.3. (a) Surface of u_h . (b) Surface of $Q_\tau u_h$.

Example 4.4: Let the domain $\Omega = [0,1] \times [0,1]$. Also, the real solution is assumed to be

$$u = \cos(xy)$$

h	$\ u - u_h\ _1$	$\ u - Q_\tau u_h\ _1$
2^{-3}	0.3023e-2	0.1428e-2
3^{-3}	0.1282e-2	0.4368e-3
4^{-3}	0.6573e-3	0.1775e-3
5^{-3}	0.3805e-3	0.8544e-4
$O(h^k), k =$	0.9966	1.3543

Table 4.4: Errors on meshes \mathcal{T}_h and \mathcal{T}_τ .

From the data shown in Table 4.4, it is clear that the exact solution u in the H_1 -norm has the superconvergence, (see Figure 4.7). Figure 4.8 (a) and (b), shows that the approximate solutions u_h and $Q_\tau u_h$. This is in agreement with the previously stated theory.

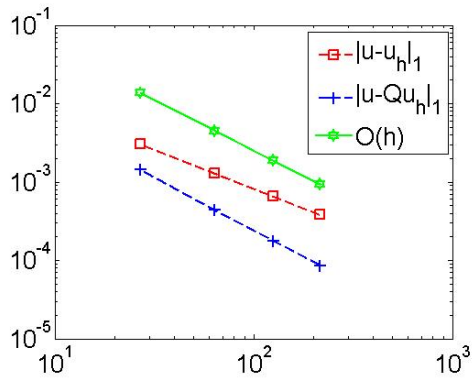


Figure 4.7: Convergence rate of error in Example 4.4. H_1 -norm error.

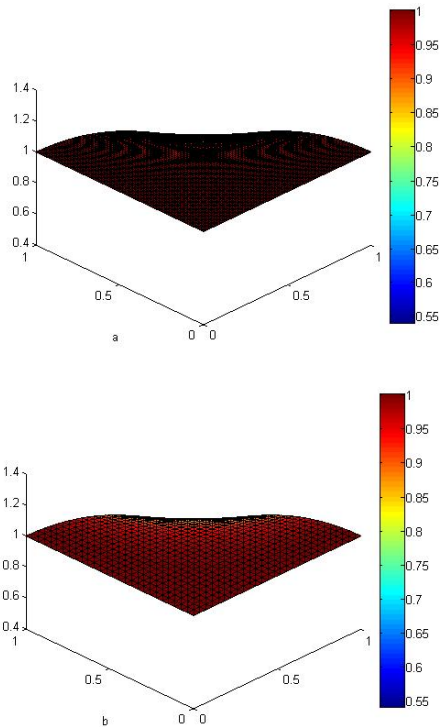


Figure 4.8: Results for $u = \cos(xy)$ in Example 4.4. (a) Surface u_h . (b) Surface of $Q_\tau u_h$.



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