*On Existence and uniqueness to solutions nonlocal integrodifferential equations*

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**ABSTRACT**

The Study aims in this paper to give and investigate the existence and uniqueness of mild solutions to nonlinear functional integrodifferential equations in Banach Spaces. the fixed point theorem, according to Sadovskii and sutible necessary conditions, are concepts consulted to obtain the results in the work .

## 1. INTRODUCTION:

Neutral differential equations has been an active field of investigation because of their applications in technical sciences, physics, and so on. In many areas of there has been wide interest in studing of functional differential equations merging memory.

previous works on existence and uniqueness of different types to solutions for differential and functional differential equations with nonlocal conditions, we refer to Byszewiski and Lakshmikantham see [4], also Lin and Liu [6].

In this work, firstly, we shall using theorem of fixed point, contraction Banach and theory of evolution families to prove and investigated solvability of mild solution to system (3.1)

Secondly, we proved uniqueness of mild solution to the problem (3.1), by using Gronwall's inequality.

## 2. PRELIMINARIES:

In this section we will take areal banach Space (Y,||.||) , where on $[0.T]$. Y is a Continuous space .

Here we give some basic concepts which will be needed in the work.

**Definition (2.1), [2]:**

"Let $X = [X, d]$ and $Y = (y, d)$ be a metric spaces. A mapping $T: X \to Y$ is said to be continuous at a point $x_0 \in X$ if for every $\in > 0$ there is a $\delta > 0$ such that $d(T_x, T_{x_o}) < \in$ for all X satisfying $d(x, x_o) < \delta$, and T is said be continuous if it is continuous at every point of X."

**Definition (2.2), [7]:**

"Let $X$ and $Y$ be normed spaces, the operator $A: D(A) \subseteq X \to Y$ is said to be compact if

**1.** A is continuous.

**2.** A transforms bounded subset M of X in to relatively compact subset in y $\left(\left(\overline{A(M)}\right) is\ compact\right)$."

**Definition (2.3), [8]:**

"A subset $U$ of C $[a, b]$ is said to be equicontinuous if for each $\in > 0$ .

There is $\delta > 0$ such that $\|x - y\| < \delta$ and $u \in U$, imply $\|u(x) - u(y)\| < \in$ ."

**Theorem (2.4), (Arzela - Ascoli's theorem), [5]:**

"Let $F \subset$C $([o, b]; X)$ satisfy:

i) For any t $\in$ [a,b], {f(t): f $\in \mathcal{F}$} is relatively compact in X.

ii) $\mathcal{F}$ is equicontinuous on [a,b], that is, for any $\in > 0$ and any $t \in [a, b]$, there exit $\delta > 0$ such that, $\|f(t) - f(s)\| < \in$, for any $\delta \in [a, b]$ satisfying $|t - s| < \delta$, and all $f \in \mathcal{F}$. Then $\mathcal{F}$ is relatively compact."

**Definition (2.5), [2]:**

"Let $X = [X,d]$ be a metric spaces. A mapping $T: X \to Y$ is called a contraction on X if there is a positive real number $\propto < 1$, such that for all $x, y \in X, d(T_x, T_x) \leq \propto d(x,y)$."

**Definition (2.6), [3]:**

"Let $X$ be a metric spaces. Equipped with a distance d. A mapping $f: X \to X$ is said to be Lipschitz continuous if there is $\lambda \geq 0$ such that: $d\big(f(x_1), f(x_2)\big) \leq \lambda\, d\,(x_1, x_2)$, for all $x_1, x_2 \in X$."

**lemma (2.7), "completely continuous [3]:**

"Let $X$ be a normed space, the mapping $A: D(A) \subseteq X \to X$ is called completely continuous if it is both Continuous and compact."

**Theorem (2.8), "Sadovskii's fixed point theorem" [1]:**

"Let P be a condensing operator on a banach space X, (that is P is a continuous and takes bounded sets into bounded sets), and $\propto P(B) \leq \propto P(B)$ for every bounded set B of $x$ with $\propto (B) > o$, if $P\big(C([o,T];X)\big) \sqsubset (C[o,T];X)$ for convex, closed and bounded set $C([o,T];X)$, then P has a bounded point in $C([o,T];X)$."

**3. EXISTENCE OF MILD SOLUTION:**

By using Y of a Banach Spaces we take the system:

$$\frac{d}{dt}\Big[u(\mathrm{t}) + K\Big(t, u(\mathrm{t}), u\big(b_1(\mathrm{t})\big), \cdots u\big(b_n(\mathrm{t})\big)\Big)\Big]$$
$$= B(t)u(\mathrm{t}) + F\Big(\mathrm{t}, u(\mathrm{t}), u\big(a_1(\mathrm{t})\big), \cdots u\big(a_{n^*}(t)\big)\Big)$$
$$+ H\left(\mathrm{t}, u(\mathrm{t}), \int_0^t h\,(s, \tau, u, (\tau))ds\right)$$

$$u(0) + g(u) = u_0 \qquad (3.1)$$

Where $B = B(t)$ is generated compact semigroup of uniformly bounded linear operators $\sqcup\ (t, s)$ defined in the Banach space Y.

**Definition (3.1):**

Let $u$ (.): $[0, T] \to Y$ is a solution to system (3.1) to $u_o \in Y$, where this equation is held :

$$u(t) = \sqcup\ (t,0)\left[u_0 + K\left(0, u(0), u\left(b_1(0)\right), \dots \dots u\left(b_n(0)\right) - g(u)\right]\right.$$
$$-K\left(t, u\left(b_1(t)\right), \dots \dots u\left(b_n(t)\right)\right)$$
$$+\int_o^t \sqcup\ (t,s) B(s) K\left(s, u(s), u\left(b_1(s)\right)\right), \dots . u\left(b_n(s)\right) ds +$$
$$\int_0^t \sqcup\ (t,s) F\left(s, u(s), u\left(a_1(s)\right)\right), \dots . u\left(a_{n^*}(s)\right) ds$$
$$+\int_0^t \sqcup\ (t,s)\left[H\left(s, u(s), \int_0^s h\left(s, \tau, u(\tau)\right) d\tau\right)\right] ds. \qquad (3.2)$$

we assume the following conditions to investigate the existence of the mild solution to the problem (3.1):

**Hypotheses (3.2):**

1. A function K: $[0, t] \times Y \to Y$ is continuous function and $B(t)k$ satisfies Lipchitz condition, as

$$\|B(t)K(t, x_i) - B(t)K(s, \overline{x_\iota})\| \leq N_1 \|x_i - \overline{x_\iota}\|$$

Where $N_1 > 0$ is constant, also

$$\|B(t)K(s,x_i)\| \leq N(max\|x_i\| + 1),$$

$$\forall t \in [0,T], i = 0,1, \dots \dots \dots \dots$$

2. A function F:$[0,T] \times Y \to Y$ is satisfied

i. $F(.,x_i): [0,T] \to Y$ is strongly measurable, $\forall\ x_i \in Y$.

ii. For $r \in N^+$ there exist a function

$$f_r \in L^1([0,T]), as$$

$$\sup_{0 \leq t \leq T} \|x_i\| \leq r\|F(.,x_i)\| \leq f_r(t) and$$

$$\lim_{r \to +\infty} \frac{1}{r} \int_0^t f_r(s)ds = \delta < \infty, \qquad where\ f_r\ is\ positive.$$

3. $H: [0,T] \times Y \times Y \to Y$ is a function as:

i. $H(.,x,u): [0,T] \to X$ is strongly continuous $\forall\ x,u \in Y$.

ii. $\eta_m \in L^1([0,T])$, where $m \in N$ is any positive number, also

$$\sup_{0 \leq t \leq T} \left\| H(s,x(s)),\ \int_0^t h\ (s,\tau,x(\tau))d\tau \right\| \leq \eta_m(s) \text{ and}$$

$$\lim_{r \to +\infty} \frac{1}{m} \int_0^t f_m(s)ds = \bar{\delta} < \infty$$

4. $g \in C([o,T];Y)$ is a completely continuous as, $\|g(x)\| \leq N\|x\| \forall\ x \in C([0,T];Y)$.

5.

i. $N_1 M M^* + N_1 M^* + MT = L_0 < 1$
ii. $M^* N_1 + M\left(N + M^* N_1 + T N_1 + \delta + \bar{\delta}\right) < 1$
iii. $\|\sqcup(t,s)\| \leq M, where\ M \geq 1, also\ sup\|B^{-1}(t)\| = M^*$

**Theorem (3.3):**

If the conditions (1-5) are holds. The system (3.1) has a mild solution on $[0.T]$.

**Proof:**

Let

$$\left(t, u(t), u\left(b_1(t)\right), \ldots \ldots u\left(b_n(t)\right)\right) = \left(t, x(t)\right),$$
$$\left(t, y(t), u\left(a_1(t)\right)\right), \ldots \ldots u\left(a_{n^*}(t)\right) = \left(t, \tilde{x}(t)\right)$$

and $R = R_1 + R_2$ is an operator define by:

$$(R_z)(t) = \sqcup(t, 0)[u_0 - K(0, x_0) - g(u)]$$
$$-K\left(t, x(t)\right) + \int_0^t \sqcup(t,s) B(s) K\left(\left(s, x(s)\right)\right) ds$$
$$+ \int_0^t \sqcup(t,s) F\left(s, \tilde{x}(s)\right) ds$$
$$+ \int_0^t \sqcup(t,s) H\left(s, u(s), \int_0^s h\left(s, \tau, u(\tau)\right) d\tau\right) ds, 0 \leq t \leq T$$

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Now, we have

$$r < \|Ru_r(\mathrm{t})\| = \big\| \sqcup (t,o)\big[u_0 + K\big(o,x(0)\big) - g(u_r)\big]$$

$$-K\big(\mathrm{t},x_r(\mathrm{t})\big) + \int_0^t \sqcup (\mathrm{t},s)\mathrm{B}(s)k\big(s,x_r(s)\big)ds\|$$

$$+\left\|\int_0^{\mathrm{t}} \sqcup (\mathrm{t},s)F\big(s,\tilde{x}(s)\big)ds + \int_0^t \sqcup (\mathrm{t},s)H\big(s,u_r(s)\big),\right.$$

$$\int_0^s h\,(s,\tau,u_r(\tau))d\tau)ds\|$$

$$\leq \|\sqcup (t,0)[u_o - g(u_r) + K(o,x_r(0)]\|$$

$$+\big\|B(\mathrm{t})B^{-1}(\mathrm{t})K\big(\mathrm{t},x_r(\mathrm{t})\big)\big\| + \int_0^{\mathrm{t}} \|\sqcup (t,s)\|\,\big\|B(s)K\big(s,x_r(s)\big)\big\|ds$$

$$+\int_0^u \|\sqcup (t,s)\|\,\big\|F\big(s,\widetilde{x_r}(s)\big)\big\| +$$

$$\int_0^t \|\sqcup (\mathrm{t},s)\|\,\left\|H\left(s,u_r(s),\int_0^s h\,(s,\tau,u_r(\tau))d\tau)ds\right)\right\|$$

$$\leq M[\|u_0\| + N_r + M^*N_1(r+1)] + M^*N_1(r+1)$$

$$+MN_1(r+1)T + M\int_0^T fr(s)ds + M\int_0^T \eta_m(s)ds$$

Then

Where $(R_1 t)(t) = \sqcup(t,0)k(0,x(o))$

$$-K(t,x(t)) + \int_0^t \sqcup(t,s)B(s)K(s,x(s))ds.$$

$$(R_2 u)(t) = \sqcup(t,0)[u_0 - g(u)] + \int_0^t \sqcup(t,s)F(s,\tilde{x}(s))ds$$

$$+ \int_0^t \sqcup(t,s)H\left(s,u(s),\int_0^s h(s,\tau,u(\tau))d\tau\right)ds \quad 0 \le t \le T$$

Hence, we let

$$A_r = \{u \in C([0,T];Y): \|u(t)\| \le r, 0 \le t \le T,\ r \ \ positive\ number\}$$

Then, we have $A_r \neq \emptyset\ in\ C([0,T];Y)$ is convex closed set, and therefore there is $u_r(.) \in A_r$,and $(Ru_r \notin A_r)$ for each r.

$$\frac{r}{r} < \|Ru_r(t)\| \le \frac{M}{r}[\|u_0\| + Nr + M^*N_1(r+1)]$$

$$+\frac{M^*N_1}{r}(r+1) + \frac{MN_1}{r}(r+1)T + \frac{M}{r}\int_0^T fr(s)ds$$

$$+\frac{M}{r}\int_0^T \eta_m(s)ds.$$

$$\lim_{r\to+\infty}\frac{r}{r}\|Ru_r(t)\| \le \lim_{r\to+\infty}\left\{\frac{M}{r}[\|u_0\| + Nr + M^*N_1(r+1)]\right.$$

$$\frac{M^*N_1}{r}(r+1)+\frac{MNT}{r}(r+1)+\frac{M}{r}\int_0^T fr(s)ds+\frac{M}{r}\int_0^T \eta_m(s)ds$$

$$\leq MN + MM^*N_1 + M^*N_1 + TMN_1 + M\delta + M\bar{\delta}$$

Hence

$$M^*N_1 + M\left(N+M^*N_1 + TN_1 + \delta + \bar{\delta}\right) \geq 1$$

But this statement contradicts with condition (5)(ii), So we obtain that

$(RA_r) \subseteq A_r$ for some positive r.

To prove $R_1$ is contraction let $u_1, u_2 \in A_r$, by condition (1) and (5), we have.

$$\|(R_1u_1)(t) - (R_1u_2)(t)\| \leq \left\|\sqcup(t,0)\left[K\left(0,x_1(0)\right) - \right.\right.$$

$$\left.K\left(0,x_2(0)\right)\right\| + \left\|K\left(t,x_1(t) - K\left(t,x_2(t)\right)\right)\right\|$$

$$+\left\|\int_0^t \sqcup(t,s)\,B(s)K\left(s,x_1(s) - B(s)K\left(s,x_2(s)\right)\right)\right\|$$

$$\leq (M+1)N_1 Sup_{0\leq t\leq T}\|u_1(s) - u_2(s)\| +$$

$$MTN_1\ Sup_{0\leq t\leq T}\|u_1(s) - u_2(s)\|$$

$$\leq [N_1MM^* + N_1M^* + MT]Sup_{0\leq t\leq T}\ \|u_1(s) - u_2(s)\|$$

$$\leq L_0\, Sup_{0\leq t\leq T}\ \ \|u_1(s) - u_2(s)\|$$

$\leq L_0 \quad \|u_1(s) - u_2(s)\|$

Thus $R_1$ is contraction.

After that we prove that $R_2$ is continuous on the set $B_r$ and let $\{u_n\} \sqsubseteq A_r$ with $u_n \to u$ in $A_r$, so we have.

$$\|R_2 u_n - R_2 u\| = Sup_{0 \leq t \leq T} \|\sqcup(t,0)[u_n(0) - u(0)]$$

$$+ \int_0^t \sqcup(t,s)\left[F(s,\tilde{x}_n(s)) - F(s,\tilde{x}(s))\right]ds\Bigg\|$$

$$+Sup_{0 \leq t \leq T} \left\| \int_0^t \sqcup(t,s)\left[H\left(s, u_n(s), \int_0^t h(s,\tau,u_n(\tau))d\tau\right) ds\right)\right.$$

$$\left.-H\left(s,u(s),\int_0^t h(s,\tau,u(\tau))d\tau\right)ds\right]\Bigg\| \longrightarrow 0 \ as\ n \longrightarrow \infty$$

That means R is continuous.

Next step, let $o \leq t_1 \leq t^* \leq T, o < \in < t_1$ and prove that $\{R_2 u : u \in B_r\}$ is equicontinuous family functions, therefore.

$$\|(R_2 u)(t^*) - (R_2 u)(t_1)\| \leq \|\sqcup(t^*,0) - \sqcup(t_1,0)\| \|u_0\| + \int_0^{t-\in} \|\sqcup(t^*,0) - \sqcup(t_1,s)\| \|F(s,\tilde{x}(s))\| ds + \int_{t_{1-\in}}^{t_1} \|\sqcup(t^*,s) - \sqcup(t,s)\| \|F(s,\tilde{x}(s))\| ds + \int_{t_1}^{t^*} \|\sqcup(t^*,s)\| \|F(s,\tilde{x}(s))\| ds$$

$$+\int_0^{t_1-\epsilon}\|\sqcup(t^*,s)-\sqcup(t,s)\|\,\|Hs,u(s),\int_0^s h(s,\tau,u(\tau))d\tau)\|ds$$

$$+\int_{t_1-\epsilon}^{t_1}\|\sqcup(t^*,s)\|\,\|H(s,u(s),\int_0^s h(s,\tau,u(\tau)d\tau)\|d\tau\|ds$$

$$+\int_{t_1}^{t^*}\|\sqcup(t^*,s)\|\,\|H(s,u(s),\int_0^s h(s,\tau,u(\tau)d\tau)d\tau)\|ds \qquad (3.3)$$

Hence, we obtain that (3.3) goes to zero when $t \to t^*$ , *and* $\{\sqcup(t,s), t-s>0\}$ is Continuous because it is Compact. So $R_2$ is equicontinuous functions.

Let the $Z(t)=\{(R_2u)(t): u\in A_r\}$ and prove it is a percompact in $Y, Z(0)$ is a percompact in Y, let $0\le t\le T$ be fixed, and $0<\epsilon<t$ , so we define

$$(R_2,\epsilon)(t)=\sqcup(t,0)u(0)+\int_0^{t-\epsilon}\sqcup(t,s)F(s,\tilde{x}(s))ds$$

$$+\int_0^{t_1-\epsilon}\sqcup(t,s)H(s,u(s),\int_0^s h(s,\tau,u(\tau))d\tau)\,ds$$

$$=\sqcup(t,0)u(0)+\sqcup(t,t-\epsilon)\int_0^{t-E}\sqcup(t-\epsilon,s)F(s,\tilde{x}(s))ds$$

$$+\sqcup(t-\epsilon,s)\int_0^{t_1-\epsilon}\sqcup(t-\epsilon,s)H(s,u(s),\int_0^s h(s,\tau,u(\tau))d\tau)ds$$

we obtain the set $Z_{\in}(\mathfrak{t})$ is a precompact in Y for every $o <\in< \mathfrak{t}$ , because $\{\sqcup(\mathfrak{t}, s), \mathfrak{t} - s > 0\}$ is compact as well as $u \in A_r$ , we have

$$\|(R_2 u)(\mathfrak{t}) - (R_2, \in)(\mathfrak{t})\| \leq \int_{\mathfrak{t}-\in}^{\mathfrak{t}} \|\sqcup(\mathfrak{t}, s) F(s, \tilde{x}(s))\| ds$$

$$+ \int_{\mathfrak{t}-\in}^{\mathfrak{t}} \|\sqcup(\mathfrak{t}, s)\| \left\|H\left(s, u(s), \int_0^s h(s, \tau, u(\tau)) d\tau\right) ds\right\| \leq$$
$$M \int_{\mathfrak{t}-\in}^{\mathfrak{t}} fr(s) ds + M \int_{\mathfrak{t}-\in}^{\mathfrak{t}} \eta_m(s) ds.$$

So we get a precompact sets that are close to $Z(\mathfrak{t})$ , thus, we obtain that $R_2$ is Compact operator by theorem (2.4).

Now Form contration of $R_1$ and Compactness of $R_2$, we have that $R = R_1 + R_2$ is contiuous and compact on $A_r$, therefore we get by (2.8) a fixed point $u(.)$ on $A_r$ which is represented a solution to problem |(3.1).

## 4. <u>UNIQUENESS OF THE MILD SOLUTION:</u>

Here, we prove the solution (3.2) to the problem (3.1) is unique.

**<u>Theorem (4.1):</u>**

If the conditions, (1), (5) (iii) and the following hypotheses:

(i) $\left\|F\left(s, \bar{\bar{u}}, (s)\right), \bar{\bar{u}}\left(a_1(s)\right), \ldots . \bar{\bar{u}}\left(a_n^*(s)\right) - F\left(s, \bar{u}(s), \bar{u}\left(a_1(s)\right), \ldots \bar{u}\left(a_n^*(s)\right) \leq L^* \sum_{i=1}^n \|u_i(s) - \bar{u}_i(s)\|\right.$

$And\ \tilde{L} = max\ F(s, O, \ldots, O)$

(ii) $\left\|H\left(s, \bar{\bar{u}}(s)\right) - H\left(s, \bar{u}(s)\right)\right\| \leq \tilde{L}_1 \|\bar{\bar{u}}(s) - \bar{u}(s)\|$

$And\ \tilde{L} = max\ \|H(s,o)\|$

(iii) $\left\|h\left(s,\tau,\bar{\bar{u}}(\tau)\right) - h\left(s,\tau,\bar{u}(\tau)\right)\right\| \leq \tilde{N}\|\bar{\bar{u}}(\tau) - \bar{u}(\tau)\|$

$And\ \tilde{N} = max\ \|h(s,\tau,0)\|, \quad also\ , MTN_1 + MTL^*\ m < 1.$

are holds. Then the mild solution (3.2) is unique .

**Proof:**

Let the problem (3.1) has two local solutions , $\bar{\bar{u}}(t), \bar{u}(t)$ on the interval $[0,T]$ . we must prove $\bar{\bar{u}}(\mathrm{t}) = \bar{u}(\mathrm{t})$ . So

$$\|\bar{\bar{u}}(\mathrm{t}) - \bar{u}(\mathrm{t})\| = \| \sqcup (\mathrm{t},0)[u_0 + K(0,u(0),u(b_1(0)), \dots \dots \dots,$$

$$u\left(b_n(0) - g(u)\right] - K(\mathrm{t},\bar{\bar{u}}\left(b_1(t)\right), \dots \dots \dots, \bar{\bar{u}}\left(b_n(\mathrm{t})\right) +$$

$$\int_0^{\mathrm{t}} \sqcup (\mathrm{t},s)B(s)K\left(s,\bar{\bar{u}}(s)\right), \bar{\bar{u}}\left(b_1(s)\right), \dots \dots, \bar{\bar{u}}\left(b_n(s)\right)ds$$

$$+ \int_0^{\mathrm{t}} \sqcup (\mathrm{t},s)F\left(s,\bar{\bar{u}}(s)\right), \bar{\bar{u}}\left(a_1(s)\right), \dots \dots, \bar{\bar{u}}\left(a_{n^*}(s)\right)ds$$

$$+ \int_0^{\mathrm{t}} \sqcup (\mathrm{t},s)\left[H\left(s,\bar{\bar{u}}(s)\right), \int_0^s h(s,\tau,\bar{\bar{u}}(\tau)d\tau)\ ds\right]$$

$$- \sqcup (\mathrm{t},o)[u_0 + k(0,u(0),u(b_1(0)), \dots .\ u(b_n(0) - g(u)]$$

$$+K\left(\mathrm{t},\bar{u}\left(b_1(\mathrm{t})\right), \dots .\ \bar{u}\left(b_n(\mathrm{t})\right) -\right.$$

$$\int_0^{\mathrm{t}} \sqcup (\mathrm{t},s)B(s)\ K\left(s,\bar{u}(s)\right), \bar{u}\left(b_1(s)\right), \dots .\ \bar{u}\left(b_n(\mathrm{t})\right)ds$$

$$- \int_0^{\mathrm{t}} \sqcup (\mathrm{t},s)F\left(s,\bar{u}(s)\right), \bar{u}\left(a_1(s)\right), \dots .\ \bar{u}\left(a_{n^*}(s)\right)ds$$

$$- \int_0^{\mathrm{t}} \sqcup (\mathrm{t},s)H\left(s,\bar{u}(s)\right), \int_0^s h\left(s,\tau,\bar{u}(\tau)d\tau\right)ds\|$$

$$\leq \int_0^t \|\sqcup (t,s)\| [\|B(s)K\left(s,\bar{\bar{u}}(s)\right), \bar{\bar{u}}\left(b_1(s)\right), \ldots . \bar{\bar{u}}\left(b_n(s)\right)$$

$$-B(s)K\left(s,\bar{u}(s)\right), \bar{u}\left(b_1(s)\right), \ldots, \bar{u}\left(b_n(s)\right)\|ds]$$

$$+ \int_0^t \|\sqcup(t,s)\| \left[\|F\left(s,\bar{\bar{u}}(s)\right), \bar{\bar{u}}\left(a_1(s)\right), \ldots . \bar{\bar{u}}\left(a_{n^*}(s)\right)\right.$$

$$-F\left(s,\bar{u}(s)\right), \bar{u}\left(a_1(s)\right), \ldots ., \bar{u}\left(a_{n^*}(s)\right)\|ds]$$

$$+ \int_0^t \|\sqcup(t,s)\| \{[H\left(s,\bar{\bar{u}}(s)\right) - H(s,0) + H(s,0)\|$$

$$+ \int_0^s \|h\left(s,\tau,\bar{\bar{u}}(\tau)\right) - h(s,\tau,0) + h(s,\tau,0)\| \, d\tau]$$

$$-H\left[\left(s,\bar{u}(s)\right) - H(s,o) + H(s,0)\right\|$$

$$+ \int_0^s \left\|h\left(s,\tau,\bar{u}(\tau)\right) - h(s,\tau,0) + h(s,\tau,0)\right\| d\tau \,]\}ds$$

There fore

$$\|\bar{\bar{u}}(t) - \bar{u}(t)\| \leq MTN_1 \|\bar{\bar{u}}_i(t) - \bar{u}_i(t)\|$$

$$+L^*MT \sum_{i=1}^{n} \|\bar{\bar{u}}_i(t) - \bar{u}_i(t)\| + MT\tilde{L}$$

$$+M \int_0^T \{\tilde{L}_1 \|\bar{\bar{u}}(s) - \bar{u}(s)\| + \tilde{\tilde{L}}$$

$$+\tilde{N} \int_0^s \|\bar{\bar{u}}(\tau) - \bar{u}(\tau)\| \, d\tau + \tilde{\tilde{N}}\Big\} \, ds$$

$$\|\bar{\bar{u}}(t) - \bar{u}(t)\| \leq (MTN_1 + MTL^*m)\|\bar{\bar{u}}(t) - \bar{u}(t)\|$$

$$+MT\tilde{L} + MT\tilde{\tilde{L}} + MT\tilde{\tilde{N}} + M\int_0^T \{\tilde{L}_1\|\bar{\bar{u}}(s) - \bar{u}(s)\|$$

$$+\tilde{N}\int_0^T \|\bar{\bar{u}}(\tau) - \bar{u}(\tau)\|d\tau\}ds$$

$$\|\bar{\bar{u}}(t) - \bar{u}(t)\|(1 - MTN_1 + MTL^*m) \leq$$

$$MT\tilde{L} + MT\tilde{\tilde{L}} + MT\tilde{\tilde{N}} + M\int_0^T \{\tilde{L}_1\|\bar{\bar{u}}(s) - \bar{u}(s)\|$$

$$+\tilde{N}\int_0^T \|\bar{\bar{u}}(\tau) - \tilde{u}(\tau)\|d\tau\}ds$$

let: $\Delta_{\bar{u},\bar{\bar{u}}} = MTN_1 + MTL^*m$

Then

$$\|\bar{\bar{u}}(t) - \bar{u}(t)\| \leq \frac{MT\left(\tilde{L}+\tilde{\tilde{L}}+\tilde{\tilde{N}}\right)+M\int_0^T\left\{\tilde{L}_1\|\bar{\bar{u}}(s)-\bar{u}(s)\|+\tilde{N}\int_0^S\|\bar{\bar{u}}(\tau)-\bar{u}(\tau)\|d\tau\right\}ds}{1-\Delta_{\bar{y},\bar{\bar{y}}}}$$

Hence by Gronwall's inequality, we get $\|\bar{\bar{u}}(t) - \bar{u}(t)\| \leq 0$
Hence we obtain that the mild solution (3.2) is unique.

## REFERENCES:

[1] BN Sadovskii, "On a fixed point principle Funct Anal Appl", 1967; 1:74-6:
[2] Erwin K., "Introdution functional Analysis with Application", By John wiley and sone, 1978.
[3] Istratescu V.I, "Fixed poiwt theory" D. Reidel publishing co., Dordrecht, 1981.
[4] Lakshmi an them, V., and Byszewski L., "Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space", Appl. Anal, 40 (1991), no. 1,11-19.
[5] Lee H., A.s., "Nonlinear nonlocal Cauchy problems in Banach spaces", applied mathematic letters 18 (2005) 401-407.
[6] Liu, J.H., and Lin, Y.P "Semi linear integrodifferentia with nonlocal Cauchy problem", Nonliear Anal. 26 (1996), no. 5, 1023-1033.
[7] Siddiq, A.H., "Functional Analysis with Application", Mc Gral-Hill publishing company limited, 1986.
[8] Zeidler, E., "Applid Functional Analysis", Springer-Verlag, new york, Inc., 1995.
[9] Zlang S., "positive solutions for boundary-value problems of nonlinear fractional differential equations", Electron. J. Differential Equations.

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## الخلاصة:

هـــدف دراســـة هـــذا البحـــث إعطـــاء وتحقيـــق وجوديـــة ووحدانيـــة الحـــل المعلول للمعادلات التفاضلية التكاملية اللا خطية في فضاء بناخ.

نظريـــــة النقطـــــة الصـــــامدة طبقـــا" لسادوفســـكي وشـــــروط ضـــــرورية مناسبة تم الاستعانة بها للحصول على نتائج هذا العمل.