On the graph of partial orders

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Abstract:

Any binary relation $\sigma \subseteq X$ (where X is an arbitrary set) generates a characteristic function on the set X^2: If $(x,y)\in\sigma$, then $\sigma(x,y)=1$, otherwise $\sigma(x,y)=0$. In terms of characteristic functions on the set of all binary relations of the set X we introduced the concept of a binary of reflexive relation of adjacency and determined the algebraic system consisting of all binary relations of a set X and all unordered pairs of various adjacent binary relations. If X is finite set then this algebraic system is a graph " a graph of graphs" in this work we investigated some features of the structures of the graph $G(X)$ of partial orders.

1. Adjacency of binary relations

Definition 1.1 Let $B \doteq \{0,1\}$ –Boolean set, X – arbitrary set, and $X^2 \doteq X\times X$ – a direct product. The function $X^2 \to B$, will be called *characteristic*. Any subset $\sigma \subseteq X^2$, called a binary relation (or relation) on the set X, generates characteristic function

$$\chi_R : X^2 \to B, \quad \chi_R(x,y) \doteq \begin{cases} 1, & \text{if } (x,y)\in R, \\ 0, & \text{if } (x,y)\notin R. \end{cases}$$

Next, the function $\chi_R(\cdot,\cdot)$ $R(x,y)$. On the other hand, any characteristic function $\chi : X^2 \to B$ generates a binary relation $R_\chi \subseteq X^2$ such that $(x,y)\in R_\chi$ if $\chi(x,y)=1$. Obviously, the map

$R \to R(\cdot,\cdot)$ is a bijection between the set of binary relations and the set of characteristic functions.

On the set of 2^{X^2} all sets of binary relations on the set X we introduce a binary reflexive adjacency.

Definition 1.2 Let $X = Y \cup Z$ – the disjoint union of two subsets (allowed, that either $Y = \varnothing$ or $Z = \varnothing$). Suppose that the relation $\sigma \subseteq X^2$ such that $\sigma(x, y) = 0$ for all $(x, y) \in Y \times Z$. It generates the relation $\tau \subseteq X^2$ such that

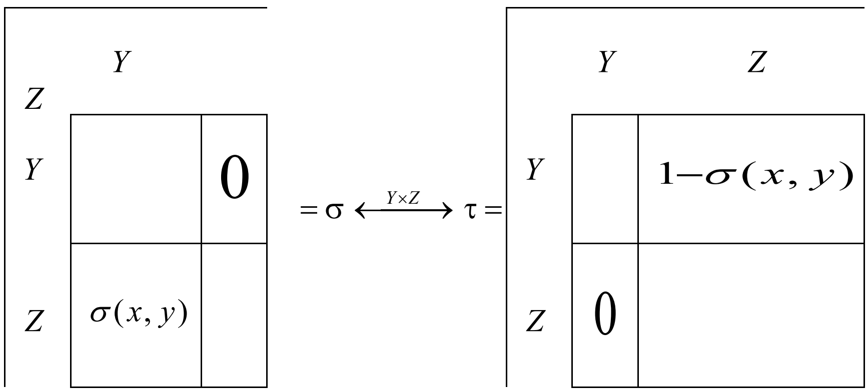
1) $\tau(x, y) = 1 - \sigma(y, x)$ for all $(x, y) \in Y \times Z$, (1)

2) $\tau(x, y) = 0$ for all $(x, y) \in Z \times Y$, (2)

3) $\tau(x, y) = \sigma(x, y)$ for all $(x, y) \in Y^2 \cup Z^2$. (3)

The relation τ is called *adjacent* with a relation σ.

Remark 1.3 From the definition it follows that if the relation τ adjacent with a relation σ, then σ adjacent with a relation τ, and this fact we write in the form of a diagram $\sigma \xleftrightarrow{Y \times Z} \tau$:



Here and elsewhere in the diagrams we mark for the value of the characteristic functions at those points which are known a priori. For example, in the block $Y \times Z$ for the relation σ we write □ *generalized* □ zero, and this means that

$\sigma(x, y) = 0$ for all $(x, y) \in Y \times Z$,

And in the same block for the relation τ we write $\mathbf{1-\sigma(x,y)}$ for all $(x, y) \in Y \times Z$.

For example, $X = \{1,\ldots,6\}$, Y={1,2}, Z={3,4,5,6},

$$\left[\begin{array}{cc|c} \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} & & \mathbf{0} \\ \hline \begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{array} & & \begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \end{array}\right] = \sigma \xleftrightarrow{Y\times Z} \tau = \left[\begin{array}{c|c} \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} & \begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array} \\ \hline \mathbf{0} & \begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \end{array}\right]$$

2. Adjacency of the partial orders

Let $V(X)$ is the collection of all partial orders set define on the set X. In the other words, the relation $\sigma \subseteq X^2$ belongs in the set $V(X)$, if satisfies the following axioms:

1) reflexivity: $(x, x) \in \sigma$;

2) transitivity: if $(x, y) \in \sigma$, $(y, z) \in \sigma$, then $(x, z) \in \sigma$;

3) antisymmetry: if $(x, y) \in \sigma$, $(y, x) \in \sigma$, then $x = y$.

In the terms of the characteristic we have: $\sigma \in V(X)$ *if and only if*

1) $\sigma(x,x)=1$ for all $x\in X$; (4)

2) $\sigma(x,y)\sigma(y,z)\le\sigma(x,z)$ for all $x,y,z\in X$; (5)

3) $\sigma(x,y)\sigma(y,x)=\delta_{xy}$ for all $x,y\in X$ (where δ_{xy} – Kronecker symbol). (6)

Theorem 2.1 *Let* σ *u* τ *– are adjacent relations(i.e* $\sigma \xleftrightarrow{Y\times Z} \tau$*). Inclusion* $\sigma\in V(X)$ *hold if and only if* $\tau\in V(X)$.

Proof. By symmetry, it suffices to prove this implication $\sigma\in V(X) \Rightarrow \tau\in V(X)$.

Let $\sigma\in V(X)$.

1. Since $\tau(x,x)=\sigma(x,x)=1$, then the reflexivity relation τ obviously.

2. Its clear that, $\tau(x,y)\tau(y,x)=\sigma(x,y)\sigma(y,x)$ for any $x,y\in X$, which proves that antisymmetry relations τ.

3. Transitivity. Let $x,z,y\in X$ such that $\tau(x,y)=\tau(y,z)=1$, in the first suppose that $y\in Y$. Since $\tau(\zeta,y)=0$ for all $\zeta\in Z$, then $x\in Y$. If $z\in Y$, then $\sigma(x,y)=\tau(x,y)=1$ and $\sigma(y,z)=\tau(y,z)=1$, and since $\sigma\in V(X)$, then $\sigma(x,z)=1$, therefore $\tau(x,z)=1$. If $z\in Z$, then $\sigma(x,y)=\tau(x,y)=1$ and $\sigma(z,y)=1-\tau(y,z)=0$, and since $\sigma\in V(X)$, then by (5) $\sigma(z,x)=\sigma(z,x)\sigma(x,y)\le\sigma(z,y)=0$, hence, $\sigma(z,x)=0$, and therefore $\tau(x,z)=1$.

Now suppose that $y\in Z$. $\tau(y,\eta)=0$ $\eta\in Y$, then $z\in Z$. If $x\in Z$, then $\sigma(x,y)=\tau(x,y)=1$ $\sigma(y,z)=\tau(y,z)=1$, $\sigma\in V(X)$, then $\sigma(x,z)=1$, and since $\sigma\in V(X)$, then by (5)

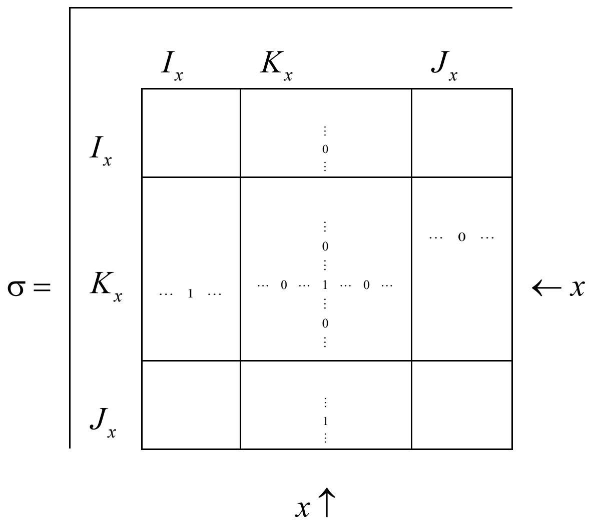
$\sigma(z,x)=\sigma(y,z)\sigma(z,x)\le\sigma(y,x)=0$, hence, $\sigma(z,x)=0$, and therefore $\tau(x,z)=1$. in all cases, we have the equality $\tau(x,z)=1$.

Thus, the set X generates a pair $\langle V(X), E(X)\rangle$, where $V(X)$ – is a set of vertices, consist of all partial orders of the set X and $E(X)$ – is a set of edges, consist of all unordered distinct pairs of adjacent partial orders of the set X. The pair $G(X)\doteq\langle V(X), E(X)\rangle$ will be called (undirected) graph of partial orders of the set X.

Definition 2.2 The partial orders σ and τ belong to the same connected component of the graph $G(X)$,if there is a finite sequence of partial orders $\sigma=\sigma_1,\sigma_2,\ldots,\sigma_m=\tau$, in which the relations σ_{k-1} and σ_k are adjacent for all $k=2,\ldots,m$. Let $G_\sigma(X)$ is the connected component of the graph $G(X)$, which contains the partial order σ.

3- On the features of the structure of the graph of partial orders.

We fix the partial order $\sigma\in V(X)$ and an element $x\in X$. For σ we have the representation:



$I_x \square I_x(\sigma) \square \{y \in X : \sigma(\mathrm{x},\mathrm{y}) = 1,\ \sigma(\mathrm{y},\mathrm{x}) = 0\},$

$\mathrm{K}_x \square K_x(\sigma) \square \{y \in X : \sigma(\mathrm{x},\mathrm{y}) = \sigma(\mathrm{y},\mathrm{x}) = \delta_{xy}\}$, Obviously, $x \in K_x$.

$J_x \square J_x(\sigma) \square \{y \in X : \sigma(\mathrm{x},\mathrm{y}) = 0,\ \sigma(\mathrm{y},\mathrm{x}) = 1\}.$

Lemma 3.1 *The following statements holds*:

1) $\sigma(y,z) = 1$ *for all* $(y,z) \in J_x \times I_x$;

2) $\sigma(y,z) = 0$ *for all* $(y,z) \in I_x \times (K_x \cup J_x)$;

3) $\sigma(y,z) = 0$ *for all* $(y,z) \in (K_x \cup I_x) \times J_x$.

Proof: Obviously, $K_x = \{y \in X : \sigma(x,y) = \sigma(y,x) = \delta_{xy}\}$,

$J_x = \{y \in X : \sigma(x,y) = 0,\ \sigma(y,x) = 1\}$,
$I_x = \{y \in X : \sigma(x,y) = 1,\ \sigma(y,x) = 0\}$.

1. Since $y \in J_x$, then $\sigma(y,x) = 1$, and since $z \in I_x$, then $\sigma(x,z) = 1$, therefore $\sigma(y,z) = 1$. In particular, $(y,z) \in I_x \times J_x$ we have the equality $\sigma(z,y) = 0$.

2. Let $(y,z) \in I_x \times K_x$.

If $z = x$, then $\sigma(y,z) = \sigma(y,x) = 0$ (since $y \in I_x$).

Let $z \neq x$, and $z \in K_x, \sigma(x,z) = 0$. Since $y \in I_x$, then $\sigma(x,y) = 1$, and then by (5) $\sigma(y,z) = \sigma(x,y)\sigma(y,z) \leq \sigma(x,z) = 0$ and therefore $\sigma(y,z) = 0$ for all $(y,z) \in I_x \times K_x$.

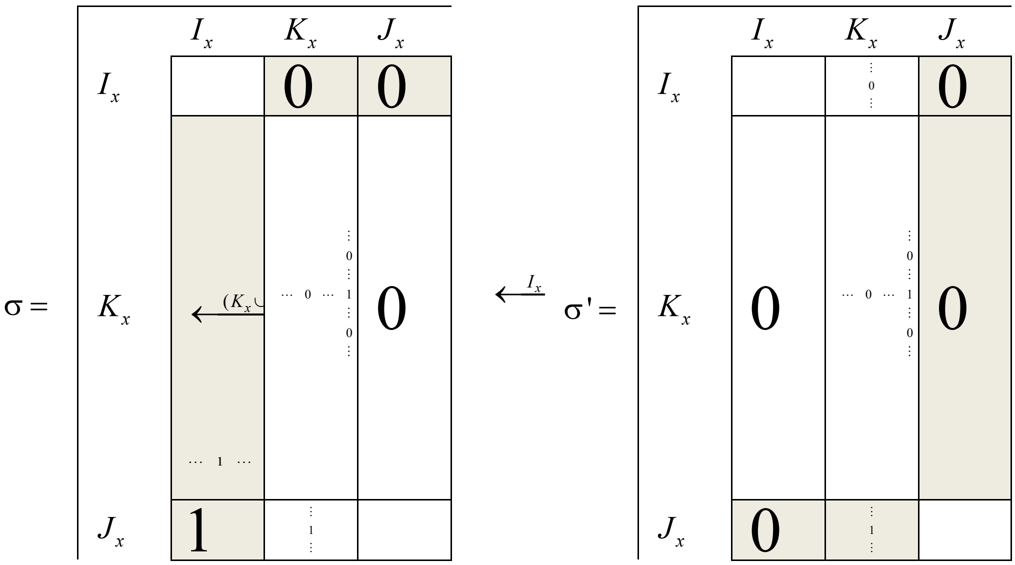
3. Let $(y,z) \in K_x \times J_x$.

If $y = x$, then $\sigma(y,z) = \sigma(x,z) = 0$ (since $z \in J_x$).

Let $y \neq x$, and $y \in K_x$, then $\sigma(y,x)=0$. Since $z \in J_x$, then $\sigma(z,x)=1$, and by (5), $\sigma(y,z)=\sigma(y,z)\sigma(z,x) \leq \sigma(y,x)=0$ therefore $\sigma(y,z)=0$ for all $(y,z) \in K_x \times J_x$

Hence we can construct a sequence of adjacent of partial orders :

$$\sigma \xleftrightarrow{I_x \times (K_x \cup J_x)} \sigma' \xleftrightarrow{(K_x \cup I_x) \times J_x} \sigma^x, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$



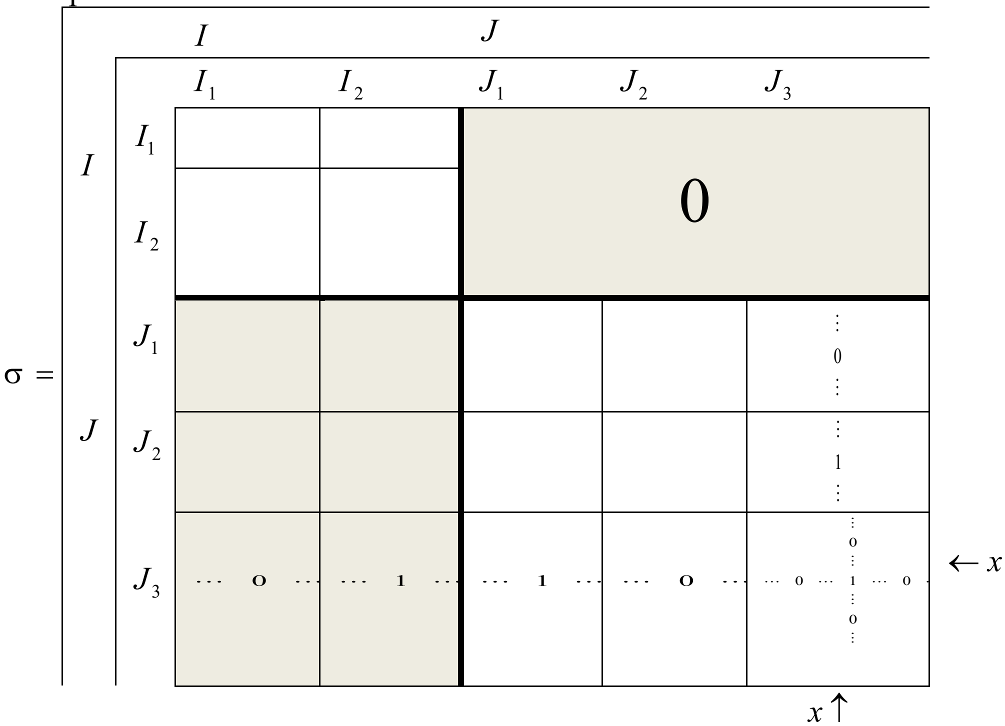
Which leads us to the partial order $\sigma^x \in V(X)$, that $\sigma^x(x,y)=\sigma^x(y,x)=\delta_{xy}$ for all $y \in X$ (in other words, if we interpret the partial order as relation $\leq$, then x is both a maximum and minimum element of a partial order σ^x).

thus, for a fixed partial order $\sigma \in V(X)$ defined a map $X \rightarrow G_\sigma(X)$, associates to an element $x \in X$ the partial order $\sigma^x \in G_\sigma(X)$ (it may be that $\sigma^x = \sigma^y$ at $x \neq y$). We also note that this map is uniquely defined in the algorithm (7) are used uniquely defined sets $I_x(\sigma)$, $K_x(\sigma)$, $J_x(\sigma)$.

Lemma 3.2 Suppose that the partial orders $\sigma, \tau \in V(X)$ belong to the same connected component of the graph $G(X)$, Then $\sigma^x = \tau^x$ for any $x \in X$.

Proof. We can assume that σ and τ – adjacent partial orders, then there is a disjoint union $I \cup J = X$ such, that : $\sigma \xleftrightarrow{I \times J} \tau$.

without loss of generality, we can also assume that $x \in J$ (if $x \in J$ in the calculations presented below the relation σ and τ changing places. For σ have the representation



$I_1 \square \{y \in I : \sigma(x,y)=0\}$,
$I_2 \square \{y \in I : \sigma(x,y)=1\}$,
$J_1 \square \{y \in J : \sigma(x,y)=1,\ \sigma(\text{y,x})=0\}$,
$J_2 \square \{y \in J : \sigma(x,y)=0,\ \sigma(\text{y,x})=1\}$,
$J_3 \square \{y \in J : \sigma(x,y)=\sigma(\text{y,x})=\delta_{xy}\}$.

Its cleary that $x \in J_3$.

1. We fix $y \in I_2 \cup J_1$, то $\sigma(x,y)=1$,since $z \in I_1$, then $\sigma(x,z)=0$ then by (5) we have $\sigma(y,z)=\sigma(x,y)\sigma(y,z) \le \sigma(x,z)=0$. Thus $\sigma(y,z)=0$ for all $(y,z) \in (I_2 \cup J_1) \times I_1$.

2. Let $(y,z) \in J_2 \times (I_2 \cup J_1)$. And since $y \in J_2$, then $\sigma(y,x)=1$, and since $z \in I_2 \cup J_1$, then $\sigma(x,z)=1$, therefore $\sigma(y,z)=1$. Thus , $\sigma(y,z)=1$ for all $(y,z) \in J_2 \times (I_2 \cup J_1)$.

3. Due to of the antisymmetry σ for all $(y,z) \in J_1 \times J_2$ have the equality $\sigma(y,z)=0$.

4. Let $(y,z) \in J_1 \times J_3$. if $z = x$, then $\sigma(y,z)=\sigma(y,x)=0$ (since $z \in J_3$).

Let $z \ne x$, and $z \in J_3$ then $\sigma(x,z)=0$. Since $y \in J_1$, then $\sigma(x,y)=1$, then from (5) $\sigma(y,z)=\sigma(x,y)\sigma(y,z) \le \sigma(x,z)=0$ .Thus , $\sigma(y,z)=0$ for all $(y,z) \in J_1 \times J_3$.

5. Let $(y,z) \in J_3 \times J_2$. If $y = x$, then $\sigma(y,z)=\sigma(x,z)=0$ (since $z \in J_2$).

Let $y \ne x$, and $y \in J_3$, то $\sigma(y,x)=0$. Since $z \in J_2$, then $\sigma(z,x)=1$, then from (5), $\sigma(y,z)=\sigma(y,z)\sigma(z,x) \le \sigma(y,x)=0$ therefore $\sigma(y,z)=0$ thus $\sigma(y,z)=0$ for all $(y,z) \in J_3 \times J_2$.

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Thus, for the adjacent of partial orders σ and τ we have the representation:

$\sigma =$

		I		J			
		I_1	I_2	J_1	J_2	J_3	
I	I_1		$*$	0	0	0	
	I_2	0		0	0	0	
J	J_1	0	$*$		0	0	
	J_2	$*$	1	1		$\vdots$ 1 $\vdots$	
	J_3	$\cdots$ 0 $\cdots$	$\cdots$ 1 $\cdots$	$\cdots$ 1 $\cdots$	0	$\vdots$ 0 $\vdots$ / $\cdots$ 0 $\cdots$ 1 $\cdots$ 0 $\cdots$ / $\vdots$ 0 $\vdots$	$\leftarrow x$
						$x \uparrow$	

$\tau =$

		I_1	I_2	J_1	J_2	J_3	
I	I_1		$*$	1	1—$*$	⋮ 1 ⋮	
	I_2	0		1—$*$	0	⋮ 0 ⋮	
J	J_1	0	0		0	0	
	J_2	0	0	1		⋮ 1 ⋮	
	J_3	0	0	··· 1 ···	0	⋮ 0 ⋮ … 0 … 1 … 0 .. ⋮ 0 ⋮	$\leftarrow x$
						$x\uparrow$	

We construct a sequence of two adjacent of partial orders:

$$\sigma \xleftrightarrow{(I_2 \cup J_1)\times(I_1 \cup J_2 \cup J_3)} \sigma' \xleftrightarrow{(I \cup J_1 \cup J_3)\times J_2} \sigma^x,$$

$$\tau \xleftrightarrow{J_1 \times (I \cup J_2 \cup J_3)} \tau' \xleftrightarrow{(I_2 \cup J_1 \cup J_3)\times(I_1 \cup J_2)} \tau^x .$$

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$\sigma =$

		I		J		
		I_1	I_2	J_1	J_2	J_3
I	I_1		*	0	0	0
	I_2	0		0	0	0
	J_1	0	*		0	0
J	J_2	*	1	1		⋮ 1 ⋮
	J_3	··· 0 ···	··· 1 ···	··· 1 ···	0	⋮ 0 ⋮ ··· 0 ··· 1 ··· 0 ⋮ 0 ⋮ $\leftarrow x$
						$x \uparrow$

$\sigma' =$

		I		J		
		I_1	I_2	J_1	J_2	J_3
I	I_1		0	0	0	0
	I_2	1—*		0	0	⋮ 0 ⋮
	J_1	1	*		0	⋮ 0 ⋮
J	J_2	*	0	0		⋮ 1 ⋮
	J_3	··· o ···	0	0	0	⋮ 0 ⋮ ··· o ··· 1 ··· o ·· ⋮ 0 ⋮ ← x
						x ↑

$\sigma^x =$

		I		J		
		I_1	I_2	J_1	J_2	J_3
I	I_1		0	0	1—*	0
	I_2	1—*		0	1	⋮ 0 ⋮
J	J_1	1	*		1	⋮ 0 ⋮
	J_2	0	0	0		0
	J_3	··· o ···	0	0	··· o ···	⋮ 0 ⋮ ··· 0 ··· 1 ··· 0 ··· ⋮ 0 ⋮ ← x
						$x \uparrow$

$\tau =$

		I_1 (I)	I_2 (I)	J_1 (J)	J_2 (J)	J_3 (J)
I	I_1		*	1	1—*	⋮ 1 ⋮
I	I_2	0		1—*	0	⋮ 0 ⋮
J	J_1	0	0		0	0
J	J_2	0	0	1		⋮ 1 ⋮
J	J_3	0	0	⋯ 1 ⋯	0	⋮ 0 ⋮ / ⋯ 0 ⋯ 1 ⋯ 0 ⋯ / ⋮ 0 ⋮ $\leftarrow x$

$x \uparrow$

$\tau' =$

		I		J		
		I_1	I_2	J_1	J_2	J_3
I	I_1			0	1—*	⋮ 1 ⋮
	I_2	0		0	0	⋮ 0 ⋮
J	J_1	0	*		0	⋮ 0 ⋮
	J_2	0	0	0		1 1 ⋮
	J_3	0	0	0	0	⋮ 0 ⋮ … 0 … 1 … 0 … ⋮ 0 ⋮ ← x

$x \uparrow$

$\tau^x =$

		I		J			
		I_1	I_2	J_1	J_2	J_3	
I	I_1		0	0	1—*	0	
	I_2	1—*		0	1	⋮ 0 ⋮	
J	J_1	1	*		1	⋮ 0 ⋮	
	J_2	0	0	0		0	
	J_3	··· 0 ···	0	0	··· 0 ···	⋮ 0 ⋮ ··· 0 ··· 1 ··· 0 ··· ⋮ 0 ⋮	$\leftarrow x$
						$x \uparrow$	

Visual comparison σ^x and τ^x shows their equality.

Corollary 3.3. *In each connected component $G_\sigma(X)$ of the graph $G(X)$ for any $x \in X$ there exists a unique $\sigma^x \in V(X)$, having the property, that $\sigma^x(x,y) = \sigma^x(y,x) = \delta_{xy}$ for all $y \in X$.*

Remark 3.4 We fix $x \in X$.
$G_\sigma(X)$ there unique partial order σ^x such that $\sigma^x(x,y) = \sigma^x(y,x) = \delta_{xy}$ $y \in X$ therefore the component $G_\sigma(X)$

tial order σ_x, define on the set $X \setminus \{x\}$, such that $\sigma_x(y,x) = \sigma^x(y,x)$ for all $y,z \in X \setminus \{x\}$.

Remark 3.5 If $card\ X < \infty$ then there exist a one - to- one between the set $V_0(X)$ and the set of all labeled of transitive graph define on the set X (see example [1, p28]) and there exist a one- to –one between these set and the set of T_0 −topology define on the set X (see example [2, p256]) and the number of these topology denoted by $T_0(n)$ and in the particular $card\ V_0(X) = T_0(n)$

Definition 3.6 For a partial order $\sigma \in V(X)$.

The set $S(\sigma) \square \{y \in X : \sigma(y,x) = \delta_{xy}\ for\ all\ x \in X\}$ is called (**support of partial order)** σ (or support set). a fact that we write in the form.

	$S(\sigma)$
$S(\sigma)$	$\begin{matrix} \ddots & & O \\ & 1 & \\ O & & \ddots \end{matrix}$
I	0
J	0

$S(G_\sigma)=\{S(\tau)\subseteq X:\tau\in G_\sigma(X)\}$ the set of support of the partial order belong to the component $G_\sigma(X)$ then:

1- $\varnothing\notin S(G_\sigma)$.

2- if $\varnothing\neq\alpha\subseteq\beta\subseteq X$ and $\beta\in S(G_\sigma)$, then $\alpha\in S(G_\sigma)$.

3- if $\alpha\subseteq X$ and $|\alpha|\le 2$, then $\alpha\in S(G_\sigma)$.

Remake 3.8 Suppose that $card$ X=n then:

1- $nT_0(n-1)$ different support sets of partial orders which contain one element.

2- $\frac{1}{2}n(n-1)T_0(n-1)$ different support sets of partial orders which contain two element.

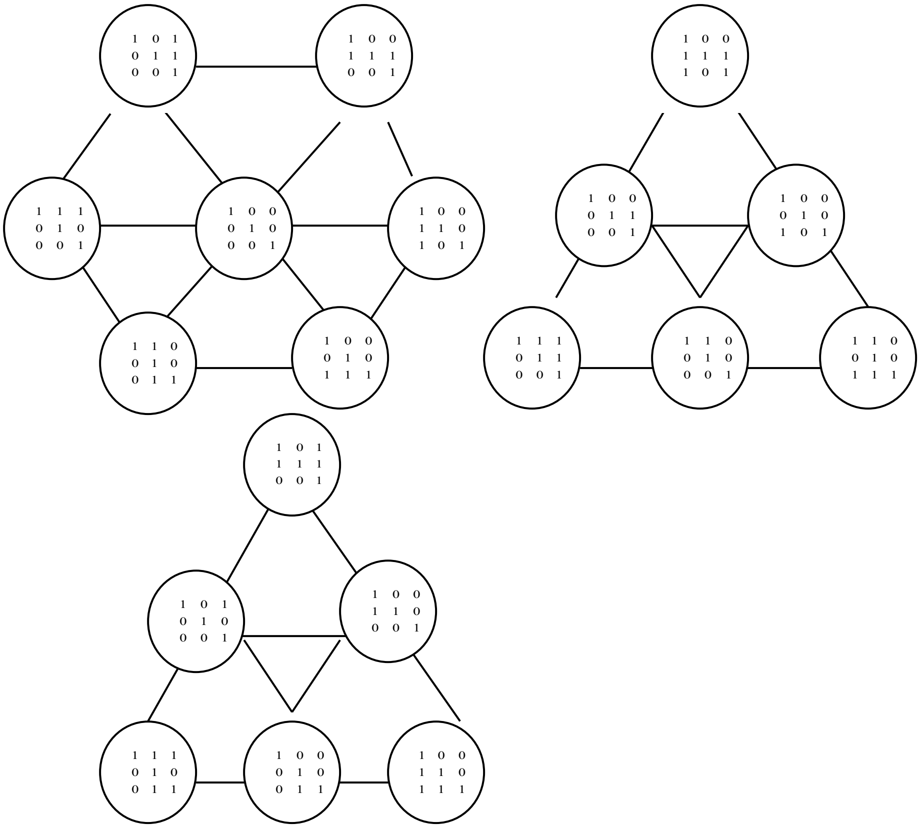
The proof of the following theorem in [3,4]

Theorem 3.9 For any $n\ge 2$ then

$$T_0(n)=\frac{1}{2}n(n+1)T_0(n-1)+card\{\sigma\in V(\{1,\dots,n\}):|S(\sigma)|\ge 3\}.$$

Example 3.10 In the graph $G(\{1,2\})$ which have unique component which contains the partial order $\begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix}\leftrightarrow\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}\leftrightarrow\begin{vmatrix}1 & 0\\ 1 & 1\end{vmatrix}$.

$G(\{1,2,3\})$ contains 19 partial order $T_0(2)=3$, and $T_0(3)=19$:



We denote the graphs of the components K_1, K_2 и K_3. It is clear that the component K_2 and K_3 are isomorphic if applied, for example, substitution $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ to the elements of the component K_2 we get the component K_3

and $S(K_1)=\{1,2,3\}$, $S(K_2)=\{\{1,2\},\{1,3\},\{2,3\}\}$, in the graph there is only one partial order, which $|S(\sigma)|\geq 3$.

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