



Some results about the restriction of Caristi's theorem to continuous function and the contraction property in Banach space

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بعض النتائج حول توسيع مبرهنة كارزتيث للدوال المستمرة وخاصية الانكماش في فضاء بناخ

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الخلاصة:

في هذه الدراسة نبين امكانية الحصول على توسيع لمبرهنة كارزتيث للدوال المستمرة بالاعتماد على مبرهنة زورملو. ومن جهة اخرى نبين خاصية الانكماش والكمال في فضاء بناخ وبعد ذلك نأخذ الكمال وخاصية الانكماش ونبين ان خاصية الانكماش تحقق الكمال في الفضاءات المترية.

ABSTRACT

.In this paper, we show the restriction of Caristi's theorem to continuous function φ can be derived directly from the Zermelo theorem., on the other hand we consider the completeness and the contraction property in Banach space and show that the contraction property implies Lipschitz-completeness or arwise-completeness in a metric spaces. However, in a metric spaces, the contraction property does not imply the usual completeness .



1-INTRODUCTION

Many authors have considered the topological characterization and the equivalence of the contraction property in fact we will show that the Caristi fixed point theorem holds only in a complete metric set and Sullivan show the same for the Ekeland principle. These lead naturally to the question whether a metric space with the contraction property is complete. This implies that a normed space is complete if and only if every contraction on the space has a fixed point, As long as the interesting result that a uniformly Lipschitz-connected (such as a convex subset of a normed space) has the contraction property if and only if the subset is complete

Let X be a non empty set and T be a self-map of X . Let $\text{Fix}(T)$ denote the set of all Fixed point of T the converse to Zermelo's fixed point theorem said that if

$\text{Fix}(T) \neq \emptyset$, then there exists a partial ordering \preceq such that every chain in (X, \preceq) has a supremum and for all $x \in X$. $x \preceq Tx$. This result is a converse of Zermelo's fixed point theorem. We also show that a normed space is complete if and only if every contraction on the space has a fixed point, As long as the interesting result that a uniformly Lipschitz-connected (such as a convex subset of a normed space) has the contraction property if and only if the subset is complete.

2-SOME PRELIMINARISS OF THE CONTRACTION

PROPERTIES

In the following there are many of more general formulations of the equivalence of coy_1, y_2 mpleteness.

Theorem 2.1.[5].

Let $\gamma : [0,1] \rightarrow C$ be Lipschitz arcs with $\gamma_1(1) = \gamma_1(0)$. Then

$$\gamma(t) = \begin{cases} \gamma_1(2t), & 0 \leq t \leq \frac{1}{2}, \\ \gamma_2(2t - 1), & \frac{1}{2} \leq t \leq 1, \end{cases}$$

is also a Lipschitz arcs .

Proof : Let $s, t \in [0,1]$. For $s, t \in \left[0, \frac{1}{2}\right]$ there exists a constant $L_1 > 0$ such that

$$d(\gamma(s), \gamma(t)) = d(\gamma_1(2s), \gamma_1(2t)) \leq L_1 |s - t|,$$

and for $s, t \in \left[\frac{1}{2}, 1\right]$, there exists a constant $L_2 > 0$ such that

$$d(\gamma(s), \gamma(t)) = d(\gamma_2(2s), \gamma_2(2t)) \leq L_2 |s - t|.$$

For $s \in \left[0, \frac{1}{2}\right]$ and $t \in \left[\frac{1}{2}, 1\right]$, we have

$$\begin{aligned} d(\gamma(s), \gamma(t)) &\leq d\left(\gamma(s), \gamma\left(\frac{1}{2}\right)\right) + d\left(\gamma\left(\frac{1}{2}\right), \gamma(t)\right) \leq L_1 \left|s - \frac{1}{2}\right| + L_2 \left|\frac{1}{2} - t\right| \\ &\leq L_1 \left(\frac{1}{2} - s\right) + L_2 \left(t - \frac{1}{2}\right) \leq L_1(t - s) + L_2(t - s) \end{aligned}$$

$$=(L_1 + L_2)|t - s|.$$

This implies g is a Lipschitz arc . ■

Theorem 2.2

Let $S = \cup \{L_\alpha : \alpha \in I\}$, where $\{L_\alpha : \alpha \in I\}$ is a family of Lipschitz-connected subset of a metric space. For arbitrary $L_\beta, L_\gamma \in \{L_\alpha : \alpha \in I\}$, there exist $L_\beta = L_0, L_1, \dots, L_k, L_{k+1} = L_\gamma$ such that $L_i \cap L_{i+1} \neq \emptyset$ for all $i=0,1,\dots,k$. Then S is Lipschitz-connected .

Proof :Assume $x_0 \in L_\beta, x_1 \in L_\gamma$, for $x_0, x_1 \in C$. There exist $L_\beta = L_0, L_1, \dots, L_k, L_{k+1} = L_\gamma$ such that $L_i \cap L_{i+1} \neq \emptyset$. By theorem (2.1) L_0, L_1 are Lipschitz-connected and $\cup_{i=0}^{k+1} L_i$ is also Lipschitz-connected and there is Lipschitz arc $x_0, x_1 \in C$ then S Lipschitz-connected . ■

Example 2.1.

Let N be a subset of a normed space defined by

$$N = \{(t, 0) : t \in [0,1] \cup (\cup_{k=0}^{\infty} \{(2^{-k}, t) ; t \in [0,1]\})\}.$$

By Theorem 2.2 N is Lipschitz-connected.

Definition 2.1[4]

Let X be a Banach space , $Y:(0,1] \rightarrow S$ is called a semi-closed arc if for each $\varepsilon > 0$ there exists some $\delta > 0$ such that $d(Y(s), Y(t)) < \varepsilon$ for all $0 \leq s < t$. a semi-closed arc Y is called Lipschitz semi-closed arc if g satisfies the Lipschitz condition .

Definition 2.2[4]

Let X be a Banachc space.

- (1) X is said to be arcwise-compete if for each semi-closed arc $y:(0,1] \rightarrow X, \lim_{s \rightarrow 0} y(s)$ exists in X ;
- (2) X is said to be Lipschitz-complete if for each Lipschitz semi-closed arc

$y:(0,1] \rightarrow X, \lim_{s \rightarrow 0} y(s)$ exists in X .

Notation.

Arcwise completeness is weaker than usual completeness even if in an arcwise connected space , so is Lipschitz-completeness . It is obvious from the definitions that Lipschitz-completeness is weaker than Arcwise completeness.

proposition 2.1[6] Let X be a Banach space .

X is arcwise-compete if $\bigcap F_n \neq \emptyset$ whenever $\{F_n: n \in N\}$ is a sequence of arcwise connected and non empty closed subset of X with $F_{n+1} \subset F_n$ and $\sup(F_n) = 0$

Proof: Let X be an arcwise-compete space. $\{F_n: n \in N\}$ is a sequence of arcwise connected and nonempty closed subset of X with $F_{n+1} \subset F_n$ and $\sup(F_n) = 0$.

For each F_n choose $x_n \in F_n$. Define g is follows:

$$y(s) = y_n(2^{n+1}s - 1), \forall \frac{1}{2^{n+1}} < s < \frac{1}{2^n}, n = 1, 2, \dots$$

Where $y_n : [0, 1] \rightarrow F_n$ such that $y_n(0) = x_{n+1}, g_n(1) = x_n$. Since $\sup(F_n) = 0$,

y is semi-closed in X and $\lim_{s \rightarrow 0} y(s) = \bar{x} \in X$ (by arcwise – completeness of X).

$y(s) \in F_n$ for all $s \in (0, \frac{1}{2^n}]$, $\bar{x} \in F_n$ (since F_n is closed) this implies that $\cap F_n \neq \emptyset$. ■

3-TWO RESTRICTION OF THE CARISTI THEOREM

the assumption of theorem (2.2)[Zermelo's theorem of fixed point]can be weakened to "each nonempty well-ordered subset has an upper bound Under the axiom of choice , "[see ,4].

The following result is independent of axiom of choice.

Theorem (3.1) [5]

The Zermelo's theorem implies the restriction of Caristi's theorem to continuous function φ . more precisely, if T is a self-map of a complete metric space (X, d) such that

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx) \text{ for all } x \in X.$$

with a continuous function $\varphi: X \rightarrow \mathbb{R}^+$, then T and X endowed with the condition

$$x \preceq y \text{ if } d(x, y) \leq \varphi(x) - \varphi(y).$$

satisfy the assumption of theorem (2.2)

proof:

Let \preceq be the partial ordering defined on X by $x \preceq y$ if $d(x,y) \leq \varphi(x) - \varphi(y)$.

by condition:

$$d(x, Tx) \leq \varphi(x) - \varphi(y) \quad \text{for all } x \in X.$$

Clearly that T is a progressive map on (X, \preceq) .

Let C be a nonempty well-ordered sub set of (X, \preceq) .

We treat C as a net, let $\{x_\sigma\}_{\sigma \in C}$ is cauchy, hence convergent to some x_0 , and $x_\sigma \preceq x_0$.

Now to show that $x_0 = \sup C$.

By the convergence we get

$$d(x_\sigma, x_0) \leq \varphi(x_\sigma) - \varphi(x_0) \quad \text{for all } \sigma \in C$$

That mean x_0 is an upper bound of C .

Let x be an arbitrary upper of C . Then

$$d(x_0, x) \leq d(x_0, x_\sigma) + d(x_\sigma, x) \leq d(x_0, x_\sigma) + \varphi(x_0) - \varphi(x)$$

So by taking limit and by continuity φ we obtain that:

$$d(x_0, x) \leq \varphi(x_0) - \varphi(x).$$

So $x_0 = \sup C$. ■

In fact the Zermelo theorem implies the Banach contraction principle.

Now to prove that the theorem (2.2) applies

Let $T: X \rightarrow X$ be a Banach contraction with a constant $\alpha \in (0,1)$.

Let $\varphi(x) = (1-\alpha)^{-1} d(x, Tx)$ for $x \in X$.

Since φ is continuous so every non-empty well-ordered subset in (X, \leq) has a supremum and T is progressive (X, \leq) . ■

Thus, the following theorem show that, when the metric is fixed, Caristi's theorem is more general than its version involving a continuous function converyed by Zermelo's theorem .

Theorem (3.2)

Let (X, d) be a metric space. If X has a limit point, then there exists a mapping T on X such that none of iterates of T Satisfies $(d(x, Tx) \leq \varphi(x) - \varphi(Tx))$ for any continuous function $\Psi: X \rightarrow \mathbb{R}$.

Proof

Let x_0 be a limit point of X . There exists a sequence $(x_n)_{n=1}^{\infty}$ such that

$x_n \rightarrow x_0, x_n \neq x_0$ and $x_i \neq x_j$ if $i \neq j$. Consider an arbitrary partition of \mathbb{N} :

$$F = \bigcup_{j \in \mathbb{N}} \{k_n^{(j)} : n \in \mathbb{N}\}$$

where $k_n^{(j)} < k_{n+1}^{(j)}$ and $k_n^{(j)} \neq k_m^{(i)}$ if $i \neq j$. For convenience denote

$$y_n^{(j)} = x_{k_n}^{(j)} \text{ for } j, n \in \mathbb{R} \text{ and } y_0^{(j)} = x_0 \text{ for } j \in \mathbb{R}.$$

Then $y_n^{(j)} \rightarrow x_0$ as $n \rightarrow$

∞ . Hence, by passing to subsequences if necessary, we may assume that for all $j, n \in \mathbb{R}$,

$$d(y_n^{(j)}, x_0) < 1 \text{ and } d(y_n^{(j)}, y_m^{(j)}) < 1/2^n \text{ for } m > n \dots\dots\dots(1)$$

We define a mapping T . Set $Tx_0 = x_0$ and for $j \in \mathbb{R}$,

$$Ty_n^{(j)} = y_{n-1}^{(j)} \text{ for } n = 1, \dots, j \text{ and } Ty_n^{(j)} = y_j^{(j)} \text{ for } n > j.$$

Denote $X_0 = \{x_0\} \cup \{y_n^{(j)} : j, n \in \mathbb{N}\}$ and set $Tx = x$ for $x \in X \setminus X_0$.

We define a function g . Set $\varphi(x_0) = 0$ and for $j \in \mathbb{R}$,

$$g(y_n^{(j)}) = \sum_{i=1}^n d(y_{i-1}^{(j)}, y_i^{(j)}) \text{ for } n = 1, \dots, j,$$

$$g(y_n^{(j)}) = \sum_{i=1}^j d(y_{i-1}^{(j)}, y_i^{(j)}) + d(y_n^{(j)}, y_j^{(j)}) \text{ for } n > j.$$

It is easily seen that by (1), for all $n \in \mathbb{R}$ and $j > 1$,

$$g(y_n^{(j)}) \leq d(x_0, y_1^{(j)}) + \sum_{i=2}^j d(y_{i-1}^{(j)}, y_i^{(j)}) + 1/2^j$$

$$< 1 + \sum_{i=2}^j 1/2^{i-1} + 1/2^j < 1 + \sum_{i=1}^{\infty} 1/2^i = 2,$$

While $g(y_1^{(j)}) = d(x_0, y_1^{(j)}) < 1$ so that $\varphi(y_n^{(j)}) < 2$ for all $j, n \in \mathbb{N}$

R. For $x \in X \setminus X_0$

, set $g(x) = 2$.

Now we show that g is lsc . Since for all $x \in X, g(x) \geq 0 = g(x_0), g$ is lower semi continuous at x_0 . Since , in fact ,

$$X_0 = \{x_n : n \in \mathbb{N}\} \cup \{x_0\} \text{ and } x_n \rightarrow x_0, \text{ we get that } X_0^d, \text{ the derived set of } X_0.$$

Since the set

$X \setminus X_0$ is open and φ is constant on it , φ is continuous at each point $x \in X \setminus X_0$.

Now suppose, there exist a $\kappa \in \mathbb{N}$ and a **continuous** function $\varepsilon: X \rightarrow \mathbb{R}$ such that (*) holds with T^κ substituted for T . Observe that for all $n > \kappa$,

$$T^\kappa y_n^{(\kappa)} = y_1^{(\kappa)} \text{ so by } (*), d(y_n^{(\kappa)}, T^\kappa y_n^{(\kappa)}) = d(y_n^{(\kappa)}, y_1^{(\kappa)}) \leq \varepsilon(y_n^{(\kappa)}) - \varepsilon(y_1^{(\kappa)}).$$

Hence taking the limit with $n \rightarrow \infty$ yields

$$d(x_0, y_1^{(\kappa)}) \leq \varepsilon(x_0) - \varepsilon(y_1^{(\kappa)}) \dots\dots\dots (2)$$

On the other hand ,by (*) we get $d(y_1^{(\kappa)}, T^\kappa y_1^{(\kappa)}) = d(y_1^{(\kappa)}, x_0) \leq \varepsilon(y_1^{(\kappa)}) - \varepsilon(x_0)$.

Hence and by (2) we obtain that $y_1^{(\kappa)} = x_0$, which yields a contradiction . ■

4- THE EQUIVALENT BETWEEN COMPLETENESS AND THE CONTRACTION PROPERTY.

In this section , we consider the equivalence relation between completeness and the contraction principle in the sense of the equivalence metric .

Theorem 4.1 [8] Let T operate on a metric space (X, ρ) . There exists abounded metric σ uniformly equivalent to ρ on X such that T is a Banach contraction under σ if and only if T is uniformly continuous and $\text{diam}(T^n X) \rightarrow 0$.

Theorem 4.2 [2] Let (X, d) be a Banach space. If X has the strong contraction property, then X is arcwise-complete.

Proof: Let \bar{X} be the completion of X $\forall (0,1] \rightarrow X$ be a semi-closed arc and

$\bar{x} = \lim_{s \rightarrow 0} \forall(s)$. Set $\forall(0) = \bar{x}$ and $X' = X \cup \{\bar{x}\}$. Then g continuously extends to a mapping on $[0, 1]$.

Observe that $\forall(s) \rightarrow \bar{x}$ as $s \rightarrow 0$. For each number sequence $\{\epsilon_k\}_{k=1}^{\infty}$ with $1 = \epsilon_0 > \epsilon_1 > \epsilon_2 > \dots$ and $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$, there exists a sequence $\{\delta_k\}_{k=1}^{\infty}$ such that $0 \leq \delta_{k+1} < \delta_k \leq \epsilon_k$ and

$$(4,1) \quad d(\forall(t')\forall(t'')) < \epsilon_k, \quad \forall 0 \leq t', t'' < \delta_k .$$

Let $\bar{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}$ and $h: X' \rightarrow [0, 1]$ be given by

$$h(x) = \begin{cases} \frac{\delta_{k+1}}{2^{k+1}} \cdot \frac{\bar{d}(x, \bar{x}) - \epsilon_{k-1}}{\epsilon_k - \epsilon_{k-1}} + \frac{\delta_k}{2^k} \cdot \frac{\epsilon_k - \bar{d}(x, \bar{x})}{\epsilon_k - \epsilon_{k-1}}, & \epsilon_k < \bar{d}(x, \bar{x}) \leq \epsilon_{k-1}, k = 1, 2, \dots \\ 0, & x = \bar{x} \end{cases}$$

Define $T: X' \rightarrow X'$ by

$$T = g \circ h.$$

each $x \in X$, we have

$$|h(x) - 0| = \left| \frac{\delta_{k+1}}{2^{k+1}} \cdot \frac{\bar{d}(x, \bar{x}) - \epsilon_{k-1}}{\epsilon_k - \epsilon_{k-1}} + \frac{\delta_k}{2^k} \cdot \frac{\epsilon_k - \bar{d}(x, \bar{x})}{\epsilon_k - \epsilon_{k-1}} \right| \leq \frac{\delta_{k+1}}{2^{k+1}} + \frac{\delta_k}{2^k} < \delta_k$$

By (4,1)

$$d(Tx, \bar{x}) = d(\forall(h(x))), \forall(0) < \epsilon_k$$

Assume $\epsilon_{k'+1} < \bar{d}(Tx, \bar{x}) \leq \epsilon_{k'}$ where $k' \geq k$. Then

$$|h(Tx) - 0| = \left| \frac{\delta_{k'+2}}{2^{k'+2}} \cdot \frac{\bar{d}(Tx, \bar{x}) - \epsilon_{k'}}{\epsilon_{k'+1} - \epsilon_{k'}} + \frac{\delta_{k'+1}}{2^{k'+1}} \cdot \frac{\epsilon_{k'+1} - \bar{d}(Tx, \bar{x})}{\epsilon_{k'+1} - \epsilon_{k'}} \right| \leq \frac{\delta_{k'+2}}{2^{k'+2}} + \frac{\delta_{k'+1}}{2^{k'+1}} < \delta_{k'+1}$$

By (4.1) $d(T^2x, \bar{x}) = d(\forall(h(Tx), g(0))) < \epsilon_{k+1}$.

Hence $d(T^n x, \bar{x}) < \epsilon_k(n-1) \leq \epsilon_n, \forall x \in X'$.

We get $\text{diam}(T'(X')) \rightarrow 0$ as $n \rightarrow \infty$ and the conditions of Theorem 4.1 are satisfied.

By Theorem 4.1, there is a metric d^* uniformly equivalent to d such that $T: X' \rightarrow X'$ is a Banach contraction mapping with respect to d^* . Thus \bar{x} is the unique fixed point of T in X' . Finally observe that $T: X \rightarrow X$ and X has the strong contraction property. ■

Theorem 4.3 [9] Let (X, d) be a locally arcwise-connected metric. If X is arcwise-complete, then X has the strong contraction property.

Proof: Let d^* be a metric uniformly equivalent to d , $T: X \rightarrow X$ a metric contraction under d^* with contraction constant $0 \leq h < 1$.

Note that T is contractive and d^* is uniformly equivalent to d . Select, as we may, x_0 in X with $d(x_0, Tx_0) < \delta$ and arc $g_0: [0, 1] \rightarrow X$ which connect x_0 and Tx_0 . For each $k \in \mathbb{N}$, define $g_k: [0, 1] \rightarrow X$ by $g_k(s) = T^k g_0(s)$ for all $s \in [0, 1]$. Then g_k connects $x_k = T^k(x_0)$ and $x_{k+1} = T^{k+1}(x_0)$ for each $k \in \mathbb{N}$. Now $\{T^n x_0\}$ is a Cauchy sequence under d^* , so that it is also a Cauchy sequence under d .

Define $g: (0, 1] \rightarrow X$ by

$$k=1,2,\dots g(s) = g_k(2^{k+1}s - 1), \quad \forall \frac{1}{2^{k+1}} < s \leq \frac{1}{2^k},$$

since T is a metric contraction, it follows that g is a semi-closed arc, $g(s) \rightarrow x^* \in X$ as $s \rightarrow 0$ and $\{T^n x_0\} \rightarrow x^*$. Finally, observe that $\{T^n x_0\}$ is a Cauchy sequence under d . Then x^* is the fixed point of T . ■

Theorem 4.3 can be considered as a generalization of the Banach contraction principle under weaker completeness.

Combining Theorem 4.2 and 4.3, we have the following theorems.

Theorem 4.4 Let X be a locally arcwise-connected metric space. Then X is arcwise-complete if and only if X has the strong contraction property.



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